Nonlinear, singular SPDE with gradient-type Gaussian noise driving infinitesimal vector field actions

joint work with Ioana Ciotir (INSA Rouen, Normandie Université)

Jonas Tölle

(Aalto University)

SPDE and Applications - X, Levico Terme

May 30–June 4, 2016

CIRM
CENTRO INTERNAZIONALE
PER LA RICERCA MATEMATICA

Aalto University
School of Science

Contents

Motivation: The linear case

Zakaï equation

Consider the following linear SPDE in a separable Hilbert space H, $t \in [0, T]$:

$$dX_t = AX_t\, dt + \sum_{i=1}^{N} B_i X_t\, d\beta_t^i, \quad X_0 = x \in H, \tag{1}$$

for β^i, $i = 1, \dots, N$, independent Brownian motions on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$.

A, B_i, $i = 1, \dots, N$ are unbounded linear operators on H.

Sufficient conditions on A and the B_i such that there exist strong solutions to (1) are given e.g. in [Da Prato & Zabczyk, Cambridge Univ. Press (1992), Chapter 6.5].

Following [Da Prato, Iannelli & Tubaro, Univ. Padova (1982)], [Da Prato, Iannelli & Tubaro, Stochastics (1982)], one obtains strong solutions to (1), whenever

- A generates a C_0-semigroup e^{tA}.
- The B_i, $i = 1, \dots, N$ generate *mutually commuting* C_0-groups e^{tB_i}, $t \in \mathbb{R}$, $i = 1, \dots, N$.
- For every $i = 1, \dots, N$, $D(B_i^2) \supset D(A)$, and $\bigcap_{i=1}^N D((B_i^*)^2)$ is a dense subset of H.
- The operator $C := A - \frac{1}{2}\sum_{i=1}^N B_i^2$, $D(C) := D(A)$ generates a C_0-semigroup e^{tC}, $t \geqslant 0$.

Compare also with [Tubaro, Stoch. Anal. Appl. (1988)], where the commutation assumption was removed using Kunita's method of *stochastic characteristics*.

Idea

Set

$$U_t := \prod_{i=1}^{N} e^{\beta_t^i B_i}, \quad t \in [0, T],$$

and let Y_t be the solution to the time-dependent random PDE

$$\frac{d}{dt} Y_t = U_t^{-1} C U_t Y_t, \quad Y_0 = x. \tag{2}$$

Then $X_t := U_t Y_t$, $t \geqslant 0$ is a solution to (1).

Note that:

- We can $\mathbb{P}$-a.s. find strong solutions Y to (2) such that $t \mapsto Y_t$ is predictable.
- In this case, X (as above) takes values in $D(C)$ $\mathbb{P} \otimes dt$-a.s. and is a strong solution to (1).

Example (strongly elliptic case)

Let $H = L^2(\mathbb{R}^d)$, $d \in \mathbb{N}$ and $N = 1$. Set

$$Ay := \sum_{i,j=1}^{d} a_{i,j}\partial_i\partial_j y + \sum_{i=1}^{d} q_i\partial_i y + ry, \quad y \in H^2(\mathbb{R}^d) =: D(A),$$

$$By = \sum_{i=1}^{d} b_i\partial_i y + cy, \quad y \in D(B) = \{y \in L^2(\mathbb{R}^d) \mid By \in L^2(\mathbb{R}^d)\}.$$

Assume for simplicity that $a_{i,j}$, q_i, r, b_i, c are all C^3 and bounded with bounded derivatives up to order 3. Assume that there exists a constant $\gamma > 0$ such that

$$\sum_{i,j=1}^{d}(a_{i,j} - \frac{1}{2}b_i b_j)\lambda_i\lambda_j \geqslant \gamma \sum_{i=1}^{d} \lambda_i^2,$$

for all $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{R}^d$. Then the above conditions are satisfied.

Can we extend this method?

Both the commutation assumption and the ellipticity of $C := A - \frac{1}{2}\sum_{i=1}^{N} B_i^2$ seem very restrictive.

Some observations:

- In view of the Trotter product formula, the commutation seems natural.
- $\frac{1}{2}\sum_{i=1}^{N} B_i^2 X\, dt$ is exactly the Itô-Stratonovich correction term of $\sum_{i=1}^{N} B_i X\, d\beta^i$.
- One could also define the B_i weakly in a Gelfand triple $H^1 \subset L^2 \subset (H^1)^*$.
- The transformation in (2) also makes sense, when the operator A (C resp.) is nonlinear, see [Barbu, Brzeźniak, Hausenblas & Tubaro, Stoch. Processes Appl. (2013)], [Barbu & Röckner, J. Eur. Math. Soc. (2015)]!
- One has to ensure that $(U_t)_{t\geqslant 0}$ is in some sense compatible with A.

Framework and the SPDE

Stochastic singular p-Laplace equations

Consider the nonlinear Stratonovich SPDE in $L^2(\mathcal{O})$,

$$dX_t = \operatorname{div}[\Psi(\nabla X_t)]\,dt + \sum_{i=1}^{N}\sum_{j=1}^{d} b_i^j \partial_j X_t \circ d\beta_t^i, \quad X_0 = x \in L^2(\mathcal{O}), \tag{3}$$

where $\mathcal{O} \subset \mathbb{R}^d$, $d \geqslant 2$ is an open bounded convex C^3-smooth domain and $b : \overline{\mathcal{O}} \to \mathbb{R}^{N\times d}$ is a C^2-smooth "coefficient field". Assume Neumann boundary conditions.

β^i, $i = 1, \ldots, N$, are independent Brownian motions on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$.

For reasons made clear later, we shall also assume that

- $\Psi = \partial(\frac{1}{p}|\cdot|^p)$, where $p \in [1,2]$. However, let us first assume for simplicity that $p \in (1,2)$.
- The "row operators" $\langle b_i, \nabla\cdot\rangle$ commute mutually (or $N = 1$).
- The "row operators" $\langle b_i, \nabla\cdot\rangle$ commute (weakly) with the Neumann Laplace.

Commutation

Let b_i be some row of b. Define the "row operators"

$$B_i u := \sum_{j=1}^{d} b_i^j \partial_j u, \quad u \in H^1(\mathcal{O}).$$

By [Sumitomo, Hokkaido Math. J. (1972)], it is necessary and sufficient for B_i to commute with the Laplace-Beltrami operator on smooth functions that

$$b_i \quad \text{is a Killing vector field,}$$

meaning that, the Jakobian of b_i is skew-symmetric, i.e.

$$\partial_j b_i^k + \partial_k b_i^j = 0 \quad \forall 1 \leqslant j, k \leqslant d. \tag{4}$$

This automatically implies that $\operatorname{div} b_i = 0$.

Theorem

Assume that $b_i \in C^2$, $1 \leqslant i \leqslant N$, with

1. $\langle b_i, \nu \rangle = 0$ *on* $\partial\mathcal{O}$ *for all* i*, where* ν *denotes the outer normal on* $\partial\mathcal{O}$*,*
2. $N = 1$ *or* $\langle b_i, \nabla b_j^l \rangle = \langle b_j, \nabla b_i^l \rangle$ *for* $i \neq j$ *and all* $1 \leqslant l \leqslant d$ *on* $\overline{\mathcal{O}}$*,*
3. Db_i *is skew-symmetric for all* i *on* $\overline{\mathcal{O}}$*.*

Then B_i *leaves Neumann boundary conditions invariant (on a core) for every* i*.*

Also, for every $u \in H^1(\mathcal{O})$

$$B_i J_\delta u = J_\delta B_i u \quad \forall 1 \leqslant i \leqslant N \ \ \forall \delta > 0. \tag{Comm}$$

Here, $J_\delta = (\mathrm{Id} - \delta\Delta)^{-1}$ denotes the resolvent of the Neumann Laplace $-\Delta$.

Shigekawa's result

Theorem ([Shigekawa, Acta Appl. Math. (2000)])

Fix $1 \leqslant i \leqslant N$. *(Comm) is implied by the following: Suppose there exists a linear subspace* $\mathcal{D} \subset \mathrm{dom}(-\Delta)$ *such that the following conditions hold:*

1. $\Delta(\mathcal{D}) \subseteq \mathrm{dom}(B_i)$,
2. $B_i(\mathcal{D}) \subseteq \mathrm{dom}(-\Delta)$,
3. $\mathcal{D}$ *is a core for* $(-\Delta, \mathrm{dom}(-\Delta))$,
4. $\mathrm{dom}(-\Delta) \subseteq \mathrm{dom}(B_i)$ *and* $\mathrm{dom}(-\Delta) \subseteq \mathrm{dom}(B_i^*)$,
5. *for any* $u \in \mathcal{D}$, *it holds that*

$$B_i \Delta u = \Delta B_i u.$$

The p-Laplace operator

With $\Psi = \partial(\frac{1}{p}|\cdot|^p)$, in particular, with "$\partial$", we denote the Gâteaux differential for $p > 1$ (the subdifferential, resp., for $p = 1$), that is, $\partial(\frac{1}{p}|\cdot|^p)(\xi) = |\xi|^{p-2}\xi$, $\xi \in \mathbb{R}^d$. We will discuss the case of $p = 1$ later.

The quasi-linear partial differential operator "$u \mapsto \mathrm{div}[\Psi(\nabla u)]$" is called *p-Laplace* and, in particular, *singular p-Laplace*, if $p < 2$.

Its negative is an extension (in the sense of monotone graphs) of $F : H^1(\mathcal{O}) \to (H^1(\mathcal{O}))^*$

$$F(u)(v) := \int_{\mathcal{O}} \langle \Psi(\nabla u(\xi)), \nabla v(\xi) \rangle \, d\xi.$$

In fact, F is the Gâteaux differential of the convex functional

$$\Phi(u) := \frac{1}{p} \int_{\mathcal{O}} |\nabla u|^p \, d\xi, \quad u \in H^1(\mathcal{O}).$$

We see that the 2-Laplace is just the Neumann Laplace operator.

Example

Let $d = 3$. Let $\mathcal{O} = B_1(0)$ be the 3D unit ball. Let $N = 1$. Let $b(\xi) = (\xi_3 - \xi_2, \xi_1 - \xi_3, \xi_2 - \xi_1)$ and denote $1 := (1, 1, 1)$ (clearly, $b(\xi) = \xi \times 1$). Then b is a Killing vector field and (3) becomes

$$\begin{cases} dX_t \in \operatorname{div}\left[\Psi\left(\nabla X_t\right)\right] dt + \langle \xi \times \nabla X_t, 1\rangle \circ d\beta_t, & \text{in } (0,T) \times \mathcal{O}, \\ X_0 = x, & \text{in } \mathcal{O}, \\ \dfrac{\partial X_t}{\partial \nu} = 0, & \text{on } (0,T) \times \partial\mathcal{O}. \end{cases}$$

Example

Above, we can take $b : \xi \mapsto \xi \times \zeta_0$ for any $\zeta_0 \in \mathbb{R}^3 \setminus \{0\}$. This is the infinitesimal generator of $SO(3)$, where ζ_0 spans the axis of rotation.

In 2D, we can take the unit disk and $b : (\xi_1, \xi_2) \mapsto (\xi_2, -\xi_1)$, generating $SO(2)$.

Example

Let $N = d$ and $b_i^j = \delta_{i,j}$, $1 \leqslant i, j \leqslant d$. Then the above conditions are satisfied on $\mathbb{T}^d$ and (3) reduces to

$$\begin{cases} dX_t \in \operatorname{div}\left[\Psi\left(\nabla X_t\right)\right] dt + \langle \nabla X_t, \circ d\beta_t \rangle, & \text{in } (0,T) \times \mathcal{O}, \\ X_0 = x, & \text{in } \mathcal{O}, \\ \dfrac{\partial X_t}{\partial \nu} = 0, & \text{on } (0,T) \times \partial\mathcal{O}. \end{cases}$$

This example is relevant in mathematical image processing of binary tomography.

The vector fields b_i are infinitesimal generators of translation groups in the coordinate directions of $\mathbb{T}^d$.

Stochastic variational inequalities (SVI)

The Statonovich SPDE

$$dX_t = \operatorname{div}[\Psi(\nabla X_t)]\,dt + \sum_{i=1}^{N}\sum_{j=1}^{d} b_i^j \partial_j X_t \circ d\beta_t^i, \quad X_0 = x \in L^2(\mathcal{O}),$$

is formally equivalent to the Itô SPDE

$$dX_t = \operatorname{div}[\Psi(\nabla X_t)]\,dt + \frac{1}{2}\sum_{i=1}^{N} B_i^2 X_t\,dt + \sum_{i=1}^{N} B_i X_t\,d\beta_t^i, \quad X_0 = x \in L^2(\mathcal{O}), \tag{SPDE}$$

where $B_i^2 := -B_i^* B_i$.

Let Z be any process ("*test-process*") of the form

$$dZ_t = G_t\,dt + \frac{1}{2}\sum_{i=1}^{N} B_i^2 Z_t\,dt + \sum_{i=1}^{N} B_i Z_t\,d\beta_t^i,$$

where G is some progressively process taking values in $L^2(\mathcal{O})$ and such that Z takes values in H^1.

Heuristics for SVI

Consider $d(X - Z)$ and apply Itô's formula "formally" for $u \mapsto \frac{1}{2}\|u\|^2_{L^2(\mathcal{O})}$. We obtain,

$$\begin{aligned}\frac{1}{2}\|X_t - Z_t\|^2_{L^2} =& \frac{1}{2}\|x - Z_0\|^2_{L^2} \\ &+ \int_0^t (\mathrm{div}[\Psi(\nabla X_s)] - G_s, X_s - Z_s)_{L^2}\, ds \\ &+ \frac{1}{2}\sum_{i=1}^N \int_0^t (B_i^2(X_s - Z_s), X_s - Z_s)_{L^2}\, ds \\ &+ \sum_{i=1}^N \int_0^t (X_s - Z_s, B_i(X_s - Z_s)\, d\beta_s^i)_{L^2} \\ &+ \frac{1}{2}\sum_{i=1}^N \int_0^t \|B_i(X_s - Z_s)\|^2_{L^2}\, ds.\end{aligned}$$

However, on a formal level, we have that

$$(B_i^2(X - Z), X - Z)_{L^2} = -\|B_i(X - Z)\|^2_{L^2}.$$

Definition of SVI solutions

This motivates the following:

Definition

Let $x \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; L^2(\mathcal{O}))$, $T > 0$. An $\{\mathcal{F}_t\}$-progressively measurable process $X \in L^2([0,T] \times \Omega; L^2(\mathcal{O}))$ is called an *SVI-solution* to (SPDE) if $\Phi(X) \in L^1([0,T] \times \Omega)$ and for every $Z \in L^2([0,T] \times \Omega; H^1(\mathcal{O}))$ such that there exist $Z_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H^1(\mathcal{O}))$, $G \in L^2([0,T] \times \Omega; L^2(\mathcal{O}))$, $\{\mathcal{F}_t\}$-progressively measurable, such that the following equality holds $L^2(\mathcal{O})$, that is,

$$Z_t = Z_0 + \int_0^t G_s \, ds + \frac{1}{2} \sum_{i=1}^N \int_0^t B_i^2 Z_s \, ds + \sum_{i=1}^N \int_0^t B_i Z_s \, d\beta_s^i,$$

$\mathbb{P}$-a.s. for all $t \in [0,T]$, we have that the following variational inequality holds true:

Definition (cont'd)

$$\begin{aligned}\frac{1}{2}\mathbb{E}\|X_t - Z_t\|^2_{L^2(\mathcal{O})} + \mathbb{E}\int_0^t \Phi(X_s)\,ds \\ \leqslant \frac{1}{2}\mathbb{E}\|x - Z_0\|^2_{L^2(\mathcal{O})} + \mathbb{E}\int_0^t \Phi(Z_s)\,ds \\ - \mathbb{E}\int_0^t (G_s, X_s - Z_s)_{L^2(\mathcal{O})}\,ds,\end{aligned}$$

for almost all $t \in [0, T]$.

Moreover, if $X \in L^2(\Omega; C([0,T]; L^2(\mathcal{O})))$, we say that X is a *(time-) continuous SVI solution* to (SPDE).

Main result

The main result

Theorem ([Ciotir & T, to appear in J. Funct. Anal. (2016)])

Let $x \in L^2(\Omega, \mathcal{F}_0.\mathbb{P}; L^2(\mathcal{O}))$. Then there is a unique time-continuous SVI solution $X \in L^2(\Omega; C([0,T]; L^2(\mathcal{O})))$ to (SPDE) in the sense of the previous definition. For two SVI solutions X, Y with initial conditions $x, y \in L^2(\Omega; L^2(\mathcal{O}))$, resp., we have that

$$\operatorname*{ess\,sup}_{t\in[0,T]} \mathbb{E}\|X_t - Y_t\|^2_{L^2(\mathcal{O})} \leqslant \mathbb{E}\|x - y\|^2_{L^2(\mathcal{O})}.$$

Some Remarks

The existence of time-continuous SVI-solutions is proved via several approximation steps.

The Stratonovich type of the equation, the a priori estimate from the previous slide and the monotone structure of the drift help us to pass to the limit. We also use a Wong-Zakai type convergence result by [Barbu, Brzeźniak, Hausenblas & Tubaro, Stoch. Processes Appl. (2013)].

It is worth mentioning this paper, where similar equations in a setting with more regularity were studied using the semigroup method presented in the beginning of this talk.

Also, in [Barbu & Röckner, J. Eur. Math. Soc. (2015)], the semigroup-transformation approach is used in an infinite dimensional-noise setting. This paper also introduces a product Itô formula for the "stochastic multiplier" $U_t = e^{\sum_{i=1}^{N} \beta_t^i B_i}$. Compare also with [Munteanu & Röckner, Preprint (2016)].

$p = 1$

The above methods also work for the case of $p = 1$, that is, $\Psi = \mathrm{Sgn}$, so that we get the stochastic Stratonovich total variation flow

$$dX_t \in \mathrm{div}[\mathrm{Sgn}(\nabla X_t)]\, dt + \sum_{i=1}^{N} \sum_{j=1}^{d} b_i^j \partial_j X_t \circ d\beta_t^i, \quad X_0 = x \in L^2(\mathcal{O}),$$

which is multi-valued, since Sgn is maximal monotone only if one defines

$$\mathrm{Sgn}(0) := \overline{B}_1(0),$$

and $\mathrm{Sgn}(\xi) := \xi/|\xi|$ for $\xi \neq 0$. Hence zeros of the gradient $\nabla X = 0$ create a special situation. We have that $\mathrm{Sgn} = \partial|\cdot|$ (the subdifferential of the Euclidean norm).

$p = 1$

However, note that

- The convex energy Φ of "$u \mapsto -\operatorname{div}[\operatorname{Sgn}(\nabla u)]$", is the total variation $\|Du\|(\mathcal{O})$ of the distributional gradient measure Du, where $u \in BV(\mathcal{O})$ is a function of bounded variation in $L^1(\mathcal{O})$ (rather than $\int_{\mathcal{O}} |\nabla u| \, d\xi$, $u \in W^{1,1}(\mathcal{O})$, which is not lower semi-continuous in L^1).
- The SVI-framework is taylored for an access to multi-valued equations via the energies (SVI-solutions are even more robust under Mosco-convergence of the energies, see [Gess & T, J. Differential Equations (2016)]).

This talk was based on [Ciotir & T, to appear in J. Funct. Anal. (2016)]. [1]

[1] http://arxiv.org/abs/1507.02576

Thank you for your attention![2]

[2] jonas.tolle@aalto.fi — https://jonas-toelle.com