The set of sub-σ-algebras is a compact metric space.

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joint work with

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Contents

1 Convergence of σ-algebras

Setup

$(\Omega, \mathcal{F}, \mathbb{P})$	separable probability space, *
$\mathcal{N}$	σ-ideal of $\mathbb{P}$-zero sets in $\mathcal{F}$,
$\mathbb{F}$	collection of sub-σ-algebras of $\mathcal{F}$,
$\mathbb{F}^*$	All σ-algebras of the type $\mathcal{A} \vee \mathcal{N}$, $\mathcal{A} \in \mathbb{F}$,

$\|\cdot\|_p := \|\cdot\|_{L^p(\mathcal{F})}$, $p \in [1, \infty]$,

$\langle f, g \rangle := \mathbb{E}[fg]$ for $f, g \in L^1(\mathcal{F})$, whenever $f \cdot g \in L^1(\mathcal{F})$.

* $(\Omega, \mathcal{F}, \mathbb{P})$ separable iff $L^2(\Omega, \mathcal{F}, \mathbb{P})$ separable as a Hilbert space.

We do not assume that $\mathcal{F}$ is countably generated!

Recall

$\bigwedge_{i\in I}\mathcal{B}_i := \bigcap_{i\in I}\mathcal{B}_i$ *"meet"*

$\bigvee_{i\in I}\mathcal{B}_i := \sigma(\bigcup_{i\in I}\mathcal{B}_i)$ *"join"*

Definition

A subset $\mathcal{N}\subset\mathcal{F}$ is called σ-*ideal*, if

1. $\varnothing\in\mathcal{N}$,
2. For $A\in\mathcal{N}$ and $B\in\mathcal{F}$ with $B\subset A$, it follows that $B\in\mathcal{N}$.
3. $A_n\in\mathcal{N}$, $n\in\mathbb{N}$ implies that $\bigcup_{n\in\mathbb{N}}A_n\in\mathcal{N}$.

Setup

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Modes of convergence (of σ-algebras)

Let $\mathcal{B}_n, \mathcal{B} \in \mathbb{F}^*$, $n \in \mathbb{N}$.

Let $P_{\mathcal{A}}(f) := \mathbb{E}[f|\mathcal{A}]$, $f \in L^1(\mathcal{F})$, $\mathcal{A} \in \mathbb{F}^*$.

Skorohod J_1-convergence	$P_{\mathcal{B}_n} 1_A \to 1_A$ in $\mathbb{P}$-probability for every $A \in \mathcal{B}$,
Strong convergence	$P_{\mathcal{B}_n} 1_A \to P_{\mathcal{B}} 1_A$ in $\mathbb{P}$-probability for every $A \in \mathcal{F}$,
L^2-varying convergence	$\|P_{\mathcal{B}_n} f\|_2 \to \|P_{\mathcal{B}} f\|_2$ for every $f \in L^2(\mathcal{F})$,
Hausdorff convergence	$[\sup_{A_n \in \mathcal{B}_n} \inf_{B \in \mathcal{B}} \mathbb{P}(A_n \Delta B) + \sup_{A \in \mathcal{B}} \inf_{B_n \in \mathcal{B}_n} \mathbb{P}(A \Delta B_n)] \to 0$, †
Set-theoretic convergence	$\liminf_n \mathcal{B}_n := \bigvee_{n \geqslant 1} \bigcap_{k \geqslant n} \mathcal{B}_k = \mathcal{B} = \bigcap_{n \geqslant 1} \bigvee_{k \geqslant n} \mathcal{B}_k =: \limsup_n \mathcal{B}_n$,
Almost-sure convergence	$P_{\mathcal{B}_n} f \to P_{\mathcal{B}} f$ $\mathbb{P}$-a.s. for any $f \in L^1(\mathcal{F})$,
Monotone convergence	$\mathcal{B}_n \uparrow \bigvee_{n \geqslant 1} \mathcal{B}_n = \mathcal{B}$ or $\mathcal{B}_n \downarrow \bigcap_{n \geqslant 1} \mathcal{B}_n = \mathcal{B}$
Operator-norm convergence	$\|P_{\mathcal{B}_n} - P_{\mathcal{B}}\|_{L(L^2(\mathcal{F}), L^2(\mathcal{F}))} \to 0$.

† $A \Delta B := (A \cup B) \setminus (A \cap B)$, *"symmetric difference"*

Recall: Martingale convergence theorem

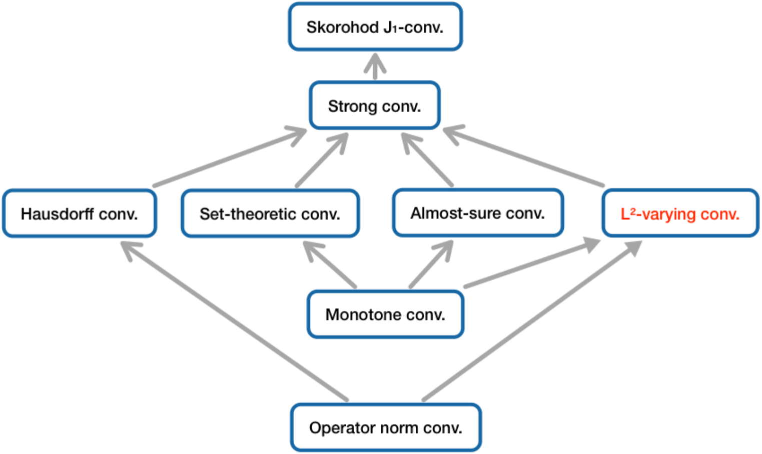
Fact

Let $\mathcal{B}_n \in \mathbb{F}^*$, $n \in \mathbb{N}$ such that $\mathcal{B}_n \subset \mathcal{B}_{n+1}$ for every $n \in \mathbb{N}$ and let $\mathcal{B} := \bigvee_{n\in\mathbb{N}} \mathcal{B}_n$. Then $\|P_{\mathcal{B}_n} f - P_{\mathcal{B}} f\|_p \to 0$ for every $f \in L^p(\mathcal{F})$, $p \in [1, \infty)$ and $P_{\mathcal{B}_n} g \to P_{\mathcal{B}} g$ $\mathbb{P}$-a.s. for any $g \in L^1(\mathcal{F})$.

Fact

Let $\mathcal{B}_n \in \mathbb{F}^*$, $n \in \mathbb{N}$ such that $\mathcal{B}_n \supset \mathcal{B}_{n+1}$ for every $n \in \mathbb{N}$ and let $\mathcal{B} := \bigcap_{n\in\mathbb{N}} \mathcal{B}_n$. Then $\|P_{\mathcal{B}_n} f - P_{\mathcal{B}} f\|_p \to 0$ for every $f \in L^p(\mathcal{F})$, $p \in [1, \infty)$ and $P_{\mathcal{B}_n} g \to P_{\mathcal{B}} g$ $\mathbb{P}$-a.s. for any $g \in L^1(\mathcal{F})$.

Implications



L^2-varying convergence of σ-algebras

Let $\mathcal{B}_n, \mathcal{B} \in \mathbb{F}^*$, $n \in \mathbb{N}$.

Let $P_{\mathcal{A}}(f) := \mathbb{E}[f|\mathcal{A}]$, $f \in L^1(\mathcal{F})$, $\mathcal{A} \in \mathbb{F}^*$.

Skorohod J_1-convergence	$P_{\mathcal{B}_n} 1_A \to 1_A$ in $\mathbb{P}$-probability for every $A \in \mathcal{B}$, ‡
Strong convergence	$P_{\mathcal{B}_n} 1_A \to P_{\mathcal{B}} 1_A$ in $\mathbb{P}$-probability for every $A \in \mathcal{F}$,
L^2-varying convergence	$\|P_{\mathcal{B}_n} f\|_2 \to \|P_{\mathcal{B}} f\|_2$ for every $f \in L^2(\mathcal{F})$,
Hausdorff convergence	$[\sup_{A_n \in \mathcal{B}_n} \inf_{B \in \mathcal{B}} \mathbb{P}(A_n \Delta B) + \sup_{A \in \mathcal{B}} \inf_{B_n \in \mathcal{B}_n} \mathbb{P}(A \Delta B_n)] \to 0$,
Set-theoretic convergence	$\liminf_n \mathcal{B}_n := \bigvee_{n \geqslant 1} \bigcap_{k \geqslant n} \mathcal{B}_k = \mathcal{B} = \bigcap_{n \geqslant 1} \bigvee_{k \geqslant n} \mathcal{B}_k =: \limsup_n \mathcal{B}_n$,
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Monotone convergence	$\mathcal{B}_n \uparrow \bigvee_{n \geqslant 1} \mathcal{B}_n = \mathcal{B}$ or $\mathcal{B}_n \downarrow \bigcap_{n \geqslant 1} \mathcal{B}_n = \mathcal{B}$
Operator-norm convergence	$\|P_{\mathcal{B}_n} - P_{\mathcal{B}}\|_{L(L^2(\mathcal{F}), L^2(\mathcal{F}))} \to 0$.

‡ *non-unique limits*

Example

Let $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}([0,1])$, i.e., the Borel σ-algebra on $[0,1]$ and $\mathbb{P} = dx \llcorner [0,1]$.

Define the sequence $\mathcal{B}_n = \sigma(I^{(n)}) \vee \mathcal{N}$, where $I^{(n)} = I(n - 2^{\lfloor \log_2(n) \rfloor}, \lfloor \log_2(n) \rfloor)$ and $I(k,m) = \left[\frac{k}{2^m}, \frac{k+1}{2^m}\right]$ for $0 \leqslant k \leqslant 2^m - 1$, $m \in \mathbb{N}$.

Example

$\mathcal{B}_n \to \{\varnothing, \Omega\} \vee \mathcal{N} =: \mathcal{B}_0$ converges L^2-varying as $n \to \infty$.

Example

There exists $g_0 \in L^2(\mathcal{F})$ such that $\mathbb{E}[g_0|\mathcal{B}_n] \not\to \mathbb{E}[g_0|\mathcal{B}_0]$ $\mathbb{P}$-a.s.

For the same g_0, there exists a (fast) subsequence $\{\mathcal{B}_{n_k}\}$ such that $\mathbb{E}[g_0|\mathcal{B}_{n_k}] \to \mathbb{E}[g_0|\mathcal{B}_0]$ $\mathbb{P}$-a.s.

Metrizability

Fact

The following metric generates the topology of L^2-varying convergence

$$d(\mathcal{A},\mathcal{B}) := \sum_{j=1}^{\infty} 2^{-j} \frac{\left| \|\mathbb{E}[f_j|\mathcal{A}]\|_2 - \|\mathbb{E}[f_j|\mathcal{B}]\|_2 \right|}{1 + \left| \|\mathbb{E}[f_j|\mathcal{A}]\|_2 - \|\mathbb{E}[f_j|\mathcal{B}]\|_2 \right|}, \quad \mathcal{A},\mathcal{B} \in \mathbb{F}^*,$$

where $\{f_j\}_{j\in\mathbb{N}}$ is a countable dense subset of $L^2(\mathcal{F})$.

Properties

Lemma

Let $\mathcal{B}_n \to \mathcal{B}$ be an L^2-varying convergent sequence of sub-σ-algebras. Then

$$\|\mathbb{E}[f|\mathcal{B}_n] - \mathbb{E}[f|\mathcal{B}]\|_2 \to 0$$

for every $f \in L^2(\mathcal{F})$, in other words, $P_{\mathcal{B}_n} \to P_{\mathcal{B}}$ in the strong operator topology.

Convergence of linear operators

Recall, $A_n, A \in L(L^2(\mathcal{F})) = L(L^2(\mathcal{F}), L^2(\mathcal{F}))$, $n \in \mathbb{N}$, converge *strongly* $A_n \to A$ as $n \to \infty$, if and only if

$$\|A_n g - Ag\|_{L^2(\mathcal{F})} \to 0,$$

for every $g \in L^2(\mathcal{F})$. Furthermore, $A_n \rightharpoonup A$ *weakly* as $n \to \infty$ if and only if

$$|\langle A_n g - Ag, h\rangle| \to 0,$$

for every $g, h \in L^2(\mathcal{F})$.

Convergence of projection operators

Let $\mathbb{S} := \{A \in L(L^2(\mathcal{F})) \ : \ \|A\|_{L(L^2(\mathcal{F}))} = 1\}$ be the *unit sphere* in $L(L^2(\mathcal{F}))$.

Let $P \in L(L^2(\mathcal{F}))$ such that $P^2 = P$ and P is self-adjoint. Then P and $\mathrm{Id} - P$ are orthogonal projections onto the closed Hilbert subspaces $\mathrm{Rg}\, P$ and $\mathrm{Ker}\, P$ respectively. If $P \not\equiv 0$, then $P \in \mathbb{S}$.

Definition (see e.g. [Attouch (1984)])

Let H be a Hilbert space. Let $C_n, C \subset H$, $n \in \mathbb{N}$, be closed, convex subsets. We say that $C_n \to C$ in *the Mosco sense*, if we have for the corresponding metric projections P_{C_n}, P_C, $n \in \mathbb{N}$, that

$$\|P_{C_n} g - P_C g\|_H \to 0$$

for any $g \in H$ as $n \to \infty$.

Convergence of projection operators

For a sequence of orthogonal projections P_n, $n \in \mathbb{N}$ and an orthogonal projection P, it holds that

$$P_n \to P \quad \text{strongly if and only if} \quad P_n \rightharpoonup P \quad \text{weakly.}$$

Fact

The unit sphere of bounded operators $\mathbb{S}$ is compact in the weak operator topology.

Caution

One assumes a priori that P is an orthogonal projection. This does not follow from the weak convergence. There are counterexamples of sequences of orthogonal projections P_n in Hilbert spaces such that $P_n \rightharpoonup Q$ and Q is not a projection.

Sub-Markovian projections

Clearly, for $\mathcal{B} \in \mathbb{F}^*$, $P_{\mathcal{B}} \not\equiv 0$, as for the one-dimensional space $L^2(\{\varnothing, \mathcal{F}\} \vee \mathcal{N}) \subset \operatorname{Rg} P_{\mathcal{B}}$.

Definition

A projection $P \in \mathbb{S}$ is called sub-Markovian, if $0 \leqslant Pf \leqslant 1$ for any $f \in L^2(\mathcal{F})$ with $0 \leqslant f \leqslant 1$.

Theorem (see e.g. [Schilling (2005)])

Let $P \in \mathbb{S}$. Then there exists $\mathcal{B} \in \mathbb{F}^*$ such that $P_{\mathcal{B}} = \mathbb{E}[\cdot|\mathcal{B}]$ if and only if P is sub-Markovian and $P1_{\Omega} = 1_{\Omega}$.

Properties

Theorem

Let $\mathcal{A}_n \to \mathcal{A}$, $\mathcal{B}_n \to \mathcal{B}$ be L^2-varying convergent sequences of sub-σ-algebras. Then

1. $\mathcal{A}_n \perp\!\!\!\perp \mathcal{B}_n$ *for every* $n \in \mathbb{N}$ *implies* $\mathcal{A} \perp\!\!\!\perp \mathcal{B}$,
2. $\mathcal{A}_n \vee \mathcal{B}_n \to \mathcal{A} \vee \mathcal{B}$ *in the* L^2*-varying sense,*
3. $\mathcal{A}_n \cap \mathcal{B}_n \to \mathcal{A} \cap \mathcal{B}$ *in the* L^2*-varying sense.*

2 Convergence in L^2-bundles

L^2-bundle spaces

Consider

$$\mathbb{H} := \bigsqcup_{\mathcal{B} \in \mathbb{F}^*} L^2(\mathcal{B})$$

i.e., *the disjoint union* of L^2-spaces, indexed by the sub-σ-algebras of $\mathcal{F}$.

The *disjoint union* is defined as the following set of pairs

$$\bigsqcup_{i \in I} A_i := \bigcup_{i \in I} \{(x, i) \mid x \in A_i\}.$$

Let

$$\pi : \mathbb{H} \to \mathbb{F}^*$$

be the (bundle) projection on the index of the element in $\mathbb{H}$.

The collection $\mathbb{H}$ mimics the total space of a fiber bundle, whereas $\mathbb{F}^*$ plays the role of the base space of a fiber bundle:

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{\ \iota\ } & L^2(\mathcal{F}) \times \mathbb{F}^* \\ {\scriptstyle \pi}\downarrow & \swarrow{\scriptstyle \mathrm{proj}_2} & \\ \mathbb{F}^* & & \end{array}$$

Here,

$$\iota : \mathbb{H} \to L^2(\mathcal{F}) \times \mathbb{F}^*, \quad \iota(u) := (u, \pi(u))$$

denotes the standard embedding and proj_2 denotes the projection on the second component.

Strong convergence

Definition (inspired by [Kuwae, Shioya (2003)])

Let $u_k, u \in \mathbb{H}$, $k \in \mathbb{N}$. We say that $u_k \to u$ *strongly* if $\pi(u_k) \to \pi(u)$ L^2-varying and there exist elements $\widetilde{u}_m \in L^2(\pi(u))$, $m \in \mathbb{N}$, such that $\|\widetilde{u}_m - u\|_2 \to 0$ as $m \to \infty$ and

$$\lim_m \limsup_k \|u_k - \mathbb{E}[\widetilde{u}_m | \pi(u_k)]\|_2 = 0.$$

Lemma

Let $u_k, u \in \mathbb{H}$, $k \in \mathbb{N}$. Suppose that $\pi(u_k) \to \pi(u)$ in the L^2-varying sense. Then the following conditions are equivalent

1. $u_k \to u$ *strongly in* $\mathbb{H}$,
2. $\lim_n \|u_k - \mathbb{E}[u|\pi(u_k)]\|_2 = 0$,
3. $\lim_k \|u_k - u\|_2 = 0$.

Weak convergence

Definition (inspired by [Kuwae, Shioya (2003)])

Let $u_k, u \in \mathbb{H}$, $k \in \mathbb{N}$. We say that $u_k \rightharpoonup u$ *weakly* if $\pi(u_k) \to \pi(u)$ in the L^2-varying sense and the following two conditions are satisfied:

1. it holds that

$$\sup_k \|u_k\|_2 < +\infty,$$

2. and, we have that

$$\lim_k \langle u_k, v_k \rangle = \langle u, v \rangle$$

for all $v_k \in L^2(\pi(u_k))$, $k \in \mathbb{N}$, $v \in L^2(\pi(u))$ such that $v_k \to v$ strongly in $\mathbb{H}$.

3 Main result

Main result§

Theorem

Suppose that $L^2(\Omega, \mathcal{F}, \mathbb{P})$ is separable. Then

$$\mathbb{F}^* = \{\mathcal{A} \vee \mathcal{N} \ : \ \mathcal{A} \text{ is a sub-}\sigma\text{-algebra of } \mathcal{F}\},$$

equipped with the topology of L^2-varying convergence, is a compact metrizable space.

Corollary

Combined with the above remarks, the result implies that the subclass of sub-Markovian projection operators P with $P1_\Omega = 1_\Omega$ is a compact subset of $\mathbb{S}$ in the strong operator topology.

§From [Beissner, T. (2017), https://arxiv.org/abs/1802.05920].

Consequence

Let $\mathcal{B}_n \in \mathbb{F}^*$, $n \in \mathbb{N}$. Consider the following property:

$$\text{There exists } \mathcal{B}_0 \in \mathbb{F}^* \text{such that } \|\mathbb{E}[u|\mathcal{B}_n] - \mathbb{E}[u|\mathcal{B}_0]\|_2 \to 0 \text{ for all } u \in L^2(\mathcal{F}) \quad \text{(E)}$$

(compare with [Tsukada, *PTRF* (1983)]).

Corollary

Let $\mathcal{B}_n \in \mathbb{F}^*$, $n \in \mathbb{N}$. Then property (E) is implied by the following: For every $u \in L^2(\mathcal{F})$, we have that

$$\exists \lim_n \|\mathbb{E}[u|\mathcal{B}_n]\|_2 \in [0, +\infty).$$

Steps of the proof

- Set $\mathbb{H}_1 := \bigsqcup_{\mathcal{B}\in\mathbb{F}^*}\{f \in L^2(\mathcal{B}) \ : \ \|f\|_2 \leqslant 1\} \subset \mathbb{H}$.
- Define a map

$$\mathcal{I} : \mathbb{H}_1 \to \bigtimes_{u\in L^2(\mathcal{F})} ([-\|u\|_2, \|u\|_2] \times [0, \|u\|_2]) =: \mathbb{T},$$

  which is verified to be:

  - injective,
  - continuous from weak to pointwise convergence,
  - with continuous inverse (pointwise-to-weak).
- The range $\mathcal{I}(\mathbb{H}_1)$ is closed[¶] in $\mathbb{T}$ and thus compact.

[¶]Limit points are in $L^2(\Sigma)$ for some $\Sigma \in \mathbb{F}^*$ — "Markov uniqueness".

Conclusion

As a consequence:

- $\mathbb{H}_1$ is compact w.r.t. the weak topology.
- As $\pi : \mathbb{H}_1 \to \mathbb{F}^*$ is continuous (weak-to-L^2-varying) and onto, $\mathbb{F}^*$ is compact!

Thank you for your attention![‖]

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