Nonlinear, singular SPDE perturbed by noise acting along infinitesimal motions on domains with symmetries

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Motivation: The linear case
Consider the following linear SPDE in a separable Hilbert space $H$, $t \in [0, T]$: \[ dX_t = AX_t \, dt + \sum_{i=1}^{N} B_i X_t \, d\beta_t^i, \quad X_0 = x \in H, \] (1)

for $\beta_i^i$, $i = 1, \ldots, N$, independent Brownian motions on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$.

$A, B_i, i = 1, \ldots, N$ are unbounded linear operators on $H$.

Sufficient conditions on $A$ and the $B_i$ such that there exist strong solutions to (1) are given e.g. in [Da Prato & Zabczyk, Cambridge Univ. Press (1992), Chapter 6.5].
Following [Da Prato, Iannelli & Tubaro, Univ. Padova (1982)], [Da Prato, Iannelli & Tubaro, Stochastics (1982)], one obtains strong solutions to (1), whenever

- $A$ generates a $C_0$-semigroup $e^{tA}$.
- The $B_i$, $i = 1, \ldots, N$ generate mutually commuting $C_0$-groups $e^{tB_i}$, $t \in \mathbb{R}$, $i = 1, \ldots, N$.
- For every $i = 1, \ldots, N$, $D(B_i^2) \supset D(A)$, and $\bigcap_{i=1}^N D((B_i^*)^2)$ is a dense subset of $H$.
- The operator $C := A - \frac{1}{2} \sum_{i=1}^N B_i^2$, $D(C) := D(A)$ generates a $C_0$-semigroup $e^{tC}$, $t \geq 0$.

Compare also with [Tubaro, Stoch. Anal. Appl. (1988)], where the commutation assumption was removed using Kunita’s method of stochastic characteristics.
Set
\[ U_t := \prod_{i=1}^{N} e^{\beta_i t} B_i, \quad t \in [0, T], \]
and let \( Y_t \) be the solution to the time-dependent random PDE
\[ \frac{d}{dt} Y_t = U_t^{-1} C U_t Y_t, \quad Y_0 = x. \]

Then \( X_t := U_t Y_t, \ t \geq 0 \) is a solution to (1).

Note that:

- We can \( \mathbb{P} \)-a.s. find strong solutions \( Y \) to (2) such that \( t \mapsto Y_t \) is predictable.

- In this case, \( X \) (as above) takes values in \( D(C) \mathbb{P} \otimes dt \)-a.s. and is a strong solution to (1).
Example (strongly elliptic case)

Let $H = L^2(\mathbb{R}^d)$, $d \in \mathbb{N}$ and $N = 1$. Set

$$Ay := \sum_{i,j=1}^{d} a_{i,j} \partial_i \partial_j y + \sum_{i=1}^{d} q_i \partial_i y + ry, \quad y \in H^2(\mathbb{R}^d) =: D(A),$$

$$By = \sum_{i=1}^{d} b_i \partial_i y + cy, \quad y \in D(B) = \{ y \in L^2(\mathbb{R}^d) \mid By \in L^2(\mathbb{R}^d) \}.$$

Assume for simplicity that $a_{i,j}$, $q_i$, $r$, $b_i$, $c$ are all $C^3$ and bounded with bounded derivatives up to order 3. Assume that there exists a constant $\gamma > 0$ such that

$$\sum_{i,j=1}^{d} (a_{i,j} - \frac{1}{2} b_i b_j) \lambda_i \lambda_j \geq \gamma \sum_{i=1}^{d} \lambda_i^2,$$

for all $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{R}^d$. Then the above conditions are satisfied.
Can we extend this method?

Both the commutation assumption and the ellipticity of \( C := A - \frac{1}{2} \sum_{i=1}^{N} B_i^2 \) seem very restrictive.

Some observations:

- In view of the Trotter product formula, the commutation seems natural.

- \( \frac{1}{2} \sum_{i=1}^{N} B_i^2 X \, dt \) is precisely the Itô-Stratonovich correction term of \( \sum_{i=1}^{N} B_i X \, d\beta^i \).

- One could also define the \( B_i \) weakly in a Gelfand triple \( H^1 \subset L^2 \subset (H^1)^* \).

- The transformation in (2) also makes sense, when the operator \( A \) (\( C \) resp.) is nonlinear, see [Barbu, Brzeźniak, Hausenblas & Tubaro, Stoch. Processes Appl. (2013)], [Barbu & Röckner, J. Eur. Math. Soc. (2015)]!

- One has to ensure that \( (U_t)_{t \geq 0} \) is in some sense compatible with \( A \).
Framework and the SPDE$^1$

$^1$based on [Ciotir & T., J. Funct. Anal. (2016)],
Consider the nonlinear Stratonovich SPDE in $L^2(\mathcal{O})$,
\[ dX_t = \text{div}[\Psi(\nabla X_t)] \, dt + \sum_{i=1}^{N} \sum_{j=1}^{d} b^i_j \partial_j X_t \circ d\beta^i_t, \quad X_0 = x \in L^2(\mathcal{O}), \quad (3) \]
where $\mathcal{O} \subset \mathbb{R}^d$, $d \geq 2$ is an open bounded convex $C^3$-smooth domain and $b : \overline{\mathcal{O}} \to \mathbb{R}^{N \times d}$ is a $C^2$-smooth “coefficient field”. Assume zero Neumann boundary conditions $\langle \nabla X, \nu \rangle = 0$, where $\nu$ denotes the outer normal on $\partial \mathcal{O}$.

$\beta^i$, $i = 1, \ldots, N$, are independent Brownian motions on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$.

We shall also assume that

- The “row operators” $\langle b_i, \nabla \cdot \rangle$ commute (weakly) with the Neumann Laplace.
- The “row operators” $\langle b_i, \nabla \cdot \rangle$ commute mutually (or $N = 1$).
- $\Psi = \partial (\frac{1}{p} | \cdot |^p)$, where $p \in [1, 2]$. However, let us first assume for simplicity that $p \in (1, 2)$. 

The noise coefficient operators

Let $b_i$ be some row of $b$. Define the “row operators”

$$B_i u := \sum_{j=1}^{d} b_i^j \partial_j u, \quad u \in H^1(\Omega).$$

By [Sumitomo, Hokkaido Math. J. (1972)], it is necessary and sufficient for $B_i$ to commute with the Laplace-Beltrami operator on smooth functions that

$$b_i \quad \text{is a Killing vector field},$$

meaning that, the Jakobian of $b_i$ is skew-symmetric, i.e.

$$\partial_j b_i^k + \partial_k b_i^j = 0 \quad \forall 1 \leq j, k \leq d. \tag{4}$$

This automatically implies that $\text{div } b_i = 0$. 
Denote by $Z^i_t : \overline{\Omega} \to \overline{\Omega}$ the one-parameter flow of diffeomorphisms on $\overline{\Omega}$ generated by $b_i$, that is
\[
\frac{d}{dt} Z^i_t = b_i(Z^i_t), \quad t \geq 0, \quad Z^i_0(\xi) = \xi \in \overline{\Omega}.
\]

Let $e^{tB_i} : L^2(\Omega) \to L^2(\Omega)$ denote the group associated to the operator $B_i u := \sum_{j=1}^{d} b^i_j \partial_j u$.

Then
\[
(e^{tB_i} u)(\xi) = u(Z^i_t(\xi))
\]
and $e^{tB_i}$ commutes with $e^{tB_j}$, $1 \leq i, j \leq N$ on smooth functions, iff $Z^i_t$ and $Z^j_t$ commute in the sense of composition of maps.

This is implied by $b^k_i \partial_k b^j_i = b^k_j \partial_k b^i_j$ on $\overline{\Omega}$ for all $1 \leq k, j \leq d$ and $i \neq l$ (or by $N = 1$).
Theorem

Assume that $b_i \in C^2$, $1 \leq i \leq N$, with

1. $\langle b_i, \nu \rangle = 0$ on $\partial \Omega$ for all $i$, where $\nu$ denotes the outer normal on $\partial \Omega$,

2. $N = 1$ or $b_i^k \partial_k b_i^j = b_i^k \partial_k b_i^i$ on $\overline{\Omega}$ for all $1 \leq k, j \leq d$ and $i \neq l$,

3. $Db_i$ is skew-symmetric for all $i$ on $\overline{\Omega}$.

Then $B_i$ leaves Neumann boundary conditions invariant (on a core) for every $i$.

Also, for every $u \in H^1(\Omega)$

$$B_i J_\delta u = J_\delta B_i u \quad \forall 1 \leq i \leq N \quad \forall \delta > 0. \quad \text{(Comm)}$$

Here, $J_\delta = (\text{Id} - \delta \Delta)^{-1}$ denotes the resolvent of the Neumann Laplace $-\Delta$. 
Shigekawa’s commutation result


Fix $1 \leq i \leq N$. (Comm) is implied by the following: Suppose there exists a linear subspace $\mathcal{D} \subset \text{dom}(-\Delta)$ such that the following conditions hold:

1. $-\Delta(\mathcal{D}) \subseteq \text{dom}(B_i)$,
2. $B_i(\mathcal{D}) \subseteq \text{dom}(-\Delta)$,
3. $\mathcal{D}$ is a core for $(-\Delta, \text{dom}(-\Delta))$,
4. $\text{dom}(-\Delta) \subseteq \text{dom}(B_i)$ and $\text{dom}(-\Delta) \subseteq \text{dom}(B_i^*)$,
5. for any $u \in \mathcal{D}$, it holds that

$$B_i \Delta u = \Delta B_i u.$$
We are able to verify 1.-5. in Shigekawa’s theorem for the core $\mathcal{D} := C^\infty(\overline{\mathcal{O}}) \cap \mathcal{C}$, where

$$\mathcal{C} := \{ u \in C^2(\overline{\mathcal{O}}) \mid \langle \nabla u, \nu \rangle = 0 \text{ on } \partial \mathcal{O} \}.$$ 

The commutation of the noise coefficient with the Neumann Laplace is needed in order to obtain a cancellation of the Itô-Stratonovich correction term in the proof of the $H^1$-energy estimate for the approximating equation

$$dX^\varepsilon_t = \text{div}[\Psi(\nabla X^\varepsilon_t)] dt + \varepsilon \Delta X^\varepsilon_t \, dt + \sum_{i=1}^{N} \sum_{j=1}^{d} b^i_j \partial_j X^\varepsilon_t \circ d\beta^i_t, \quad X^\varepsilon_0 = x \in H^1(\mathcal{O}),$$

i.e., in order to get (morally) that

$$E \| \nabla X^\varepsilon_t \|_{L^2(\mathcal{O};\mathbb{R}^d)}^2 + 2\varepsilon E \int_0^t \| \Delta X^\varepsilon_s \|_{L^2(\mathcal{O})}^2 \, ds \leq E \| \nabla x \|_{L^2(\mathcal{O};\mathbb{R}^d)}^2.$$ 

Note that the convexity of the domain is needed in order to prove

$$-\int_{\mathcal{O}} \text{div}[\Psi(\nabla X^\varepsilon)] \Delta X^\varepsilon \, d\xi \leq 0.$$
The $p$-Laplace operator

With $\Psi = \partial(\frac{1}{p}|\cdot|^p)$, in particular, with “$\partial$”, we denote the Gâteaux differential for $p > 1$ (the subdifferential, resp., for $p = 1$), that is, $\partial(\frac{1}{p}|\cdot|^p)(\xi) = |\xi|^{p-2}\xi$, $\xi \in \mathbb{R}^d$.

We will discuss the case of $p = 1$ later.

The quasi-linear partial differential operator “$u \mapsto \text{div}[\Psi(\nabla u)]$” is called $p$-Laplace and, in particular, singular $p$-Laplace, if $p < 2$.

Its negative is an extension (in the sense of monotone graphs) of $F : H^1(\mathcal{O}) \to (H^1(\mathcal{O}))^*$

$$F(u)(v) := \int_{\mathcal{O}} \langle \Psi(\nabla u(\xi)), \nabla v(\xi) \rangle \, d\xi.$$ 

In fact, $F$ is the Gâteaux differential of the convex functional

$$\Phi(u) := \frac{1}{p} \int_{\mathcal{O}} |\nabla u|^p \, d\xi, \quad u \in H^1(\mathcal{O}).$$

We see that the 2-Laplace is just the Neumann Laplace operator.
Example

Let \( d = 3 \). Let \( \mathcal{O} = B_1(0) \) be the 3D unit ball. Let \( N = 1 \). Let \( b(\xi) = (\xi_3 - \xi_2, \xi_1 - \xi_3, \xi_2 - \xi_1) \) and denote \( 1 := (1, 1, 1) \) (clearly, \( b(\xi) = \xi \times 1 \)). Then \( b \) is a Killing vector field and (3) becomes

\[
\begin{align*}
    dX_t &\in \text{div} \left[ \Psi \left( \nabla X_t \right) \right] dt + \langle \xi \times \nabla X_t, 1 \rangle \circ d\beta_t, & \text{in} \ (0, T) \times \mathcal{O}, \\
    X_0 &= x, & \text{in} \ \mathcal{O}, \\
    \frac{\partial X_t}{\partial \nu} &= 0, & \text{on} \ (0, T) \times \partial \mathcal{O}.
\end{align*}
\]

Example

Above, we can take \( b : \xi \mapsto \xi \times \zeta_0 \) for any \( \zeta_0 \in \mathbb{R}^3 \setminus \{0\} \). This is the infinitesimal generator of \( SO(3) \), where \( \zeta_0 \) spans the axis of rotation.

In 2D, we can take the unit disk and \( b : (\xi_1, \xi_2) \mapsto (\xi_2, -\xi_1) \), generating \( SO(2) \).
Example

Let $N = d$ and $b^i_j = \delta_{i,j}$, $1 \leq i, j \leq d$. Then the above conditions are satisfied on $\mathbb{T}^d$ and (3) reduces to

$$
\begin{aligned}
\left\{
\begin{array}{l}
dX_t \in \text{div} \left[ \Psi \left( \nabla X_t \right) \right] dt + \left\langle \nabla X_t, \circ d\beta_t \right\rangle, \\
X_0 = x, \\
\frac{\partial X_t}{\partial \nu} = 0,
\end{array}
\right.
\end{aligned}
$$

in $(0, T) \times \mathcal{O}$,

in $\mathcal{O}$,

on $(0, T) \times \partial \mathcal{O}$.

This example is relevant in mathematical image processing of binary tomography.

The vector fields $b_i$ are infinitesimal generators of translation groups in the coordinate directions of $\mathbb{T}^d$. 
Fact: Killing vector fields in flat space are rigid motions

In Euclidean space $M = \mathbb{R}^d$ or in the flat torus $M = \mathbb{T}^d$, every Killing vector field is the infinitesimal generator of a Lie group of elements of the form

$$\xi \mapsto A\xi + \xi_0$$

where $A \in SO(d)$ is the special orthogonal group (the orthogonal $d \times d$-matrices with $\det A = 1$) and $\xi_0 \in M$ is a translation by a vector. This group is called group of direct affine isometries or rigid motions and is denoted by $SE(d)$. These are the isometries of $M$ that preserve orientation.

Remark: $SO(d)$ is the connected component of $O(d)$ (the orthogonal group) which contains the identity. Connectedness is needed for the infinitesimal characterization.
Stochastic variational inequalities (SVI)

The Statonovich SPDE

\[ dX_t = \text{div}[\Psi(\nabla X_t)] \, dt + \sum_{i=1}^{N} \sum_{j=1}^{d} b_i^j \partial_j X_t \circ d\beta_t^i, \quad X_0 = x \in L^2(\mathcal{O}), \]

is formally equivalent to the Itô SPDE

\[ dX_t = \text{div}[\Psi(\nabla X_t)] \, dt + \frac{1}{2} \sum_{i=1}^{N} B_i^2 X_t \, dt + \sum_{i=1}^{N} B_i X_t \, d\beta_t^i, \quad X_0 = x \in L^2(\mathcal{O}), \]  

(SPDE)

where \( B_i^2 := -B_i^* B_i \).

Let \( Z \) be any process ("test-process") of the form

\[ dZ_t = G_t \, dt + \frac{1}{2} \sum_{i=1}^{N} B_i^2 Z_t \, dt + \sum_{i=1}^{N} B_i Z_t \, d\beta_t^i, \]

where \( G \) is some progressively process taking values in \( L^2(\mathcal{O}) \) and such that \( Z \) takes values in \( H^1 \).
Consider $d(X - Z)$ and apply Itô's formula “formally” for $u \mapsto \frac{1}{2} \|u\|_{L^2(O)}^2$. We obtain,

\[
\frac{1}{2} \|X_t - Z_t\|_{L^2}^2 = \frac{1}{2} \|X - Z_0\|_{L^2}^2 \\
\quad + \int_0^t (\text{div}[\Psi(\nabla X_s)] - G_s, X_s - Z_s)_{L^2} ds \\
\quad + \frac{1}{2} \sum_{i=1}^N \int_0^t (B_i^2(X_s - Z_s), X_s - Z_s)_{L^2} ds \\
\quad + \sum_{i=1}^N \int_0^t (X_s - Z_s, B_i(X_s - Z_s) d\beta_s^i)_{L^2} \\
\quad + \frac{1}{2} \sum_{i=1}^N \int_0^t \|B_i(X_s - Z_s)\|_{L^2}^2 ds.
\]

However, on a formal level, we have that

\[
(B_i^2(X - Z), X - Z)_{L^2} = -\|B_i(X - Z)\|_{L^2}^2.
\]
This motivates the following:

**Definition**

Let \( x \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; L^2(\mathcal{O})) \), \( T > 0 \). An \( \{\mathcal{F}_t\}\)-progressively measurable process \( X \in L^2([0, T] \times \Omega; L^2(\mathcal{O})) \) is called an **SVI-solution** to (SPDE) if \( \Phi(X) \in L^1([0, T] \times \Omega) \) and for every \( Z \in L^2([0, T] \times \Omega; H^1(\mathcal{O})) \) such that there exist \( Z_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H^1(\mathcal{O})) \), \( G \in L^2([0, T] \times \Omega; L^2(\mathcal{O})) \), \( \{\mathcal{F}_t\}\)-progressively measurable, such that the following equality holds \( L^2(\mathcal{O}) \), that is,

\[
Z_t = Z_0 + \int_0^t G_s \, ds + \frac{1}{2} \sum_{i=1}^N \int_0^t B^2_i Z_s \, ds + \sum_{i=1}^N \int_0^t B_i Z_s \, dB^i_s,
\]

\( \mathbb{P} \)-a.s. for all \( t \in [0, T] \), we have that the following variational inequality holds true:
Definition (cont’d)

\[
\frac{1}{2} \mathbb{E} \|X_t - Z_t\|_{L^2(\mathcal{O})}^2 + \mathbb{E} \int_0^t \Phi(X_s) \, ds \\
\leq \frac{1}{2} \mathbb{E} \|x - Z_0\|_{L^2(\mathcal{O})}^2 + \mathbb{E} \int_0^t \Phi(Z_s) \, ds \\
- \mathbb{E} \int_0^t (G_s, X_s - Z_s)_{L^2(\mathcal{O})} \, ds,
\]

for almost all \( t \in [0, T] \).

Moreover, if \( X \in L^2(\Omega; C([0, T]; L^2(\mathcal{O}))) \), we say that \( X \) is a \((time-) \) \textit{continuous SVI solution} to (SPDE).
Main result
The main result


Let $x \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; L^2(\mathcal{O}))$. Then there is a unique time-continuous SVI solution $X \in L^2(\Omega; C([0, T]; L^2(\mathcal{O})))$ to (SPDE) in the sense of the previous definition. For two SVI solutions $X, Y$ with initial conditions $x, y \in L^2(\Omega; L^2(\mathcal{O}))$, resp., we have that

$$\text{ess sup}_{t \in [0, T]} \mathbb{E}\|X_t - Y_t\|_{L^2(\mathcal{O})}^2 \leq \mathbb{E}\|x - y\|_{L^2(\mathcal{O})}^2.$$
Some Remarks

The existence of time-continuous SVI-solutions is proved via several approximation steps.

The Stratonovich type of the equation, the a priori estimate from the previous slide and the monotone structure of the drift help us to pass to the limit. We also use a Wong-Zakai type convergence result by [Barbu, Brzeźniak, Hausenblas & Tubaro, Stoch. Processes Appl. (2013)] which is in the spirit of [Brzeźniak, Capiński, Flandoli, Stochastics (1988)].

It is worth mentioning this paper, where similar equations in a setting with more regularity were studied using the semigroup method presented in the beginning of this talk, also known as the Doss–Sussmann–transformation.

Also, in [Barbu & Röckner, J. Eur. Math. Soc. (2015)], the semigroup-transformation approach is used in an infinite dimensional-noise setting. This paper also introduces a product Itô formula for the “stochastic multiplier” $U_t = e^{\sum_{i=1}^N \beta_i^t B_i}$. Compare also with [Munteanu & Röckner, Preprint, to appear in IDAQP (2016)].
The above methods also work for the case of \( p = 1 \), that is, \( \Psi = \text{Sgn} \), so that we get the stochastic Stratonovich total variation flow

\[
dX_t \in \text{div}[\text{Sgn}(\nabla X_t)] \, dt + \sum_{i=1}^{N} \sum_{j=1}^{d} b^i_j \partial_j X_t \circ d\beta^i_t, \quad X_0 = x \in L^2(\mathcal{O}),
\]

which is multi-valued, since \( \text{Sgn} \) is maximal monotone only if one defines

\[
\text{Sgn}(0) := \overline{B}_1(0),
\]

and \( \text{Sgn}(\xi) := \xi/|\xi| \) for \( \xi \neq 0 \). Hence zeros of the gradient \( \nabla X = 0 \) create a special situation. We have that \( \text{Sgn} = \partial| \cdot | \) (the subdifferential of the Euclidean norm).
Nonlinear, singular SPDE perturbed by noise acting along infinitesimal motions on domains with symmetries

Main result

$p = 1$

However, note that

- The convex energy $\Phi$ of “$u \mapsto -\text{div}[\text{Sgn}(\nabla u)]$”, is the total variation $\|Du\|(\mathcal{O})$ of the distributional gradient measure $Du$, where $u \in BV(\mathcal{O})$ is a function of bounded variation in $L^1(\mathcal{O})$ (rather than $\int_{\mathcal{O}} |\nabla u| d\xi$, $u \in W^{1,1}(\mathcal{O})$, which is not lower semi-continuous in $L^1$).

- The SVI-framework permits an access to multi-valued equations via the energies / variational potentials (SVI-solutions are quite robust under Mosco-convergence of the energies, see [Gess & T., J. Differential Equations (2016)]).
Congratulations Michael!

Thank you for your attention!²