Stochastic nonlinear PDEs with singular drift and gradient noise

Jonas Tölle

乔 涛

"Qiáo Tāo"

(Universität Augsburg)

Stochastic Systems: Their Analysis, Geometry and Perturbation

Academy of Mathematics and System Science

Chinese Academy of Sciences, Beijing

July 14, 2018

Imperial College

The p-Laplace equation

The p-Laplace evolution equation

$$\partial_t u = \operatorname{div}(|\nabla u|^{p-2} \nabla u) + f$$

is called

- *degenerate* for $2 < p < \infty$,
- *singular* for $1 < p < 2$,
- it reduces to the *heat equation* for $p = 2$,
- the **borderline case** $p = 1$ is called *total variation (TV-) flow*.

Aim: Replace the *force* f with *noise input* $G(\cdot, u, \nabla u)\,\partial_t W(t, \cdot)$.

Incompressible non-Newtonian fluid dynamics.

For $\mathrm{v}(\cdot) \in \mathbb{R}^3$, the p-Laplace type *diffusivity* can be found in

$$\begin{cases} \operatorname{div} \mathrm{v} = 0 \\ \partial_t \mathrm{v} + \operatorname{div}(\mathrm{v} \otimes \mathrm{v}) - \nu \operatorname{div}(|D\mathrm{v}|^{p-2} D\mathrm{v}) \quad = -\nabla \pi + \mathrm{b} \end{cases}$$

the equations for the velocity-field of a *power-law fluid*.

$p > 2$

Dilatant, or *shear-thickening*.

Examples: ooze / oobleck (mixture of water and corn-starch).

$p < 2$

Pseudoplastic, or *shear-thinning*.

Examples: hair-gel (polymeric molecules).

$p = 2$

Situation of *Newtonian fluids* — *Navier-Stokes equations.*

Examples: water, glycerol, ethanol.

Stochastic *singular* p-Laplace equation

Well-posedness for the stochastic **scalar** singular p-Laplace evolution equation

(with additive Gaussian noise)

$$du = \operatorname{div}(|\nabla u|^{p-2}\nabla u)\,dt + dW$$

on a bounded domain $\mathcal{O} \subset \mathbb{R}^d$ for the case $\left(1 \vee \frac{2d}{d+2}\right) < p < 2$ was first studied in

- [Liu, W., *J. Math. Anal. Appl.* (2009)]

in a similar variational setup as in

- [Zhang, X., *Stochastics and Dynamics* (2009)],
- [Ren, J./Röckner, M./Wang, F.-Y., *J. Differential Equations* (2007)],

extending [Krylov, N.V./Rozovskii, B.L. (1979)].

For stochastically forced **vector-valued** case, see e.g.: [Breit, D., *J. Math. Fluid Mech.* (2015)].

Stochastic *singular* p-Laplace equation

Further well-posedness results for multiplicative noise (**not** depending on ∇u)

$$du = \mathrm{div}(|\nabla u|^{p-2}\nabla u)\,dt + B(u)\,dW, \quad u(0) = u_0,$$

were obtained in:

- [Barbu, V./Röckner, M., *Arch. Ration. Mech. Anal.* (2013)] for $p = 1$ (TV-flow)
- [Gess, B./**T**, *J. Math. Pures Appl.* (2014)] for $p \in [1, 2]$, for initial data $u_0 \in H^1(\mathcal{O})$
- [Barbu, V./Röckner, M., *J. Eur. Math. Soc.* (2015)] general method for linear multiplicative noise
- [Gess, B./**T**, *J. Differential Equations* (2016)] for $p \in [1, 2]$, for initial data $u_0 \in L^2(\mathcal{O})$ and for **nonlocal** p-Laplace; **robustness** of the solutions for $p \to 1$
- [Gess, B./Röckner, M., *Trans. Amer. Math. Soc.* (2017)], higher regularity and more general noise
- [Marinelli, C./Scarpa, L., *SPDE: Anal. Comp.* (2018)], more general growth conditions

Stratonovich noise p-Laplace

On the d-dimensional flat torus $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$, consider:

$$dX_t = \operatorname{div}(a^*\phi(a\nabla X_t))\,dt + \langle b\nabla X_t, \circ d\beta_t\rangle, \quad X_0 = x \in L^2(\mathbb{T}^d). \tag{SPDE}$$

(SPDE) reduces to stochastic p-Laplace with multiplicative Stratonovich gradient noise/transport noise for $a = 1$ and for

$$\phi(z) = |z|^{p-2}z, \quad z \in \mathbb{R}^d,\ p \in (1,\infty).$$

Note.

Here, we shall consider the case of **any** $d \in \mathbb{N}$ and **any** $p \in [1,2]$.

$\longrightarrow$ We **do not need** *Sobolev embeddings.*

Stratonovich noise p-Laplace

On the d-dimensional flat torus $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$, consider:

$$dX_t = \operatorname{div}(a^*\phi(a\nabla X_t))\,dt + \langle b\nabla X_t, \circ d\beta_t\rangle, \quad X_0 = x \in L^2(\mathbb{T}^d),$$

with:

1. a monotone nonlinearity $\phi : \mathbb{R}^d \to \mathbb{R}^d$ or (multi-valued) $\phi : \mathbb{R}^d \to 2^{\mathbb{R}^d}$ of (at most) linear growth *(corresponds to $p \leqslant 2$)*,

Stratonovich noise p-Laplace

On the d-dimensional flat torus $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$, consider:

$$dX_t = \operatorname{div}(a^*\phi(a\nabla X_t))\,dt + \langle b\nabla X_t, \circ d\beta_t\rangle, \quad X_0 = x \in L^2(\mathbb{T}^d),$$

with:

1. a monotone nonlinearity $\phi : \mathbb{R}^d \to \mathbb{R}^d$ or (multi-valued) $\phi : \mathbb{R}^d \to 2^{\mathbb{R}^d}$ of (at most) linear growth *(corresponds to $p \leqslant 2$)*,
2. C^1-coefficient fields $a, b : \mathbb{T}^d \to \mathbb{R}^{d\times d}$,

Stratonovich noise p-Laplace

On the d-dimensional flat torus $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$, consider:

$$dX_t = \operatorname{div}(a^*\phi(a\nabla X_t))\,dt + \langle b\nabla X_t, \circ d\beta_t\rangle, \quad X_0 = x \in L^2(\mathbb{T}^d),$$

with:

1. a monotone nonlinearity $\phi : \mathbb{R}^d \to \mathbb{R}^d$ or (multi-valued) $\phi : \mathbb{R}^d \to 2^{\mathbb{R}^d}$ of (at most) linear growth *(corresponds to $p \leqslant 2$)*,
2. C^1-coefficient fields $a, b : \mathbb{T}^d \to \mathbb{R}^{d\times d}$,
3. a d-dimensional Wiener process $t \mapsto \beta_t = (\beta_t^1, \dots, \beta_t^d)$ on a filtered standard probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$.

(Evolution) variational inequalities — general idea

Suppose that the nonlinear operator

$$A(u) := \operatorname{div}(a^*\phi(a\nabla u)), \quad u \in \operatorname{dom}(A),$$

is the negative Gâteaux gradient

$$-D\Psi(\cdot) : \operatorname{dom}(D\Psi) \subset L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d)^*$$

for some (convex) potential $\Psi : L^2(\mathbb{T}^d) \to \mathbb{R}$.

We will see soon, that this is the case.

Assume first that $b \equiv 0$. Then (~~S~~PDE) is has a gradient flow structure in L^2,

$$\dot{u} = -D\Psi(u), \quad u \in L^2.$$

Evolution variational inequalities

Starting from a solution $[0, T] \ni t \mapsto u(t)$ to

$$\dot{u} = -D\Psi(u), \quad u \in L^2,$$

such that $u(0) = x$,

$$du = -D\Psi(u)\, dt$$

Evolution variational inequalities

Starting from a solution $[0,T] \ni t \mapsto u(t)$ to

$$\dot{u} = -D\Psi(u), \quad u \in L^2,$$

such that $u(0) = x$, we may write for $z \in C^1([0,T], L^2)$,

$$d(u - z) = -D\Psi(u)\,dt - \dot{z}\,dt.$$

Evolution variational inequalities

Starting from a solution $[0,T] \ni t \mapsto u(t)$ to

$$\dot{u} = -D\Psi(u), \quad u \in L^2,$$

such that $u(0) = x$, we may write for $z \in C^1([0,T], L^2)$,

$$d(u - z) = -D\Psi(u)\,dt - \dot{z}\,dt.$$

Applying the change of variable formula for $\|\cdot\|_{L^2}^2$ (where $D\|u\|_{L^2}^2 = 2(u,\cdot)_{L^2}$), we get for $t \in [0,T]$,

$$\|u(t) - z(t)\|_{L^2}^2 - \|x - z(0)\|_{L^2}^2 = 2\int_0^t (D\Psi(u(s)) + \dot{z}(s), z(s) - u(s))_{L^2}\,ds.$$

Evolution variational inequalities

Recall that Ψ was assumed convex and hence $D\Psi$ satisfies (in fact, is characterized by) its *subpotential property*

$$(D\Psi(u), z-u)_{L^2} \leqslant \Psi(z) - \Psi(u) \quad \forall z \in L^2.$$

Then

$$\|u(t)-z(t)\|_{L^2}^2 - \|x-z(0)\|_{L^2}^2 = 2\int_0^t (D\Psi(u(s)) + \dot{z}(s), z(s)-u(s))_{L^2}\, ds$$

leads to

$$\begin{aligned} &\|u(t)-z(t)\|_{L^2}^2 + 2\int_0^t \Psi(u(s))\, ds \\ \leqslant &\|x-z(0)\|_{L^2}^2 + 2\int_0^t \Psi(z(s))\, ds + 2\int_0^t (\dot{z}(s), z(s)-u(s))_{L^2}\, ds \end{aligned} \tag{EVI}$$

Evolution variational inequalities (EVI)

Definition

Let $x \in L^2$. A continuous path $u \in C([0,T], L^2)$ is called *EVI-solution* to

$$\dot{u} = -D\Psi(u), \quad u(0) = x,$$

if

$$\Psi(u(\cdot)) \in L^1([0,T]) \tag{REG}$$

and if for every $t \in [0,T]$ and for every $z \in C^1([0,T], L^2)$ it holds that

$$\begin{aligned} &\|u(t) - z(t)\|_{L^2}^2 + 2\int_0^t \Psi(u(s))\,ds \\ \leqslant &\|x - z(0)\|_{L^2}^2 + 2\int_0^t \Psi(z(s))\,ds + 2\int_0^t (\dot{z}(s), z(s) - u(s))_{L^2}\,ds. \end{aligned} \tag{EVI}$$

Some remarks

- We can dispense with the requirement that $u(\cdot) \in \mathrm{dom}(D\Psi)$ for which usually $\mathrm{dom}(D\Psi) \subsetneq L^2$ holds.
- The right hand side of (EVI) can be equal to $+\infty$.
- *Strong solutions* are automatically EVI-solutions, as seen above.
- For **sufficiently regular** Ψ, EVI-solutions are also strong solutions.

In many cases, and also in our case, $\mathrm{dom}(\Psi) \subsetneq L^2$. We overcome this by setting $\Psi \equiv +\infty$ on $L^2 \setminus \mathrm{dom}(\Psi)$.

One would also like to have that the new potential $\Psi : L^2 \to \mathbb{R} \cup \{+\infty\}$ is lower semi-continuous (l.s.c.) in L^2.

In the convex situation, one can usually take the l.s.c. envelope for this purpose.

The gradient structure of $A = \operatorname{div}(a^*\phi(a\nabla\cdot))$

Let $\phi = \partial\psi$, where $\psi : \mathbb{R}^d \to [0, \infty)$.

(For convex $\psi : \mathbb{R}^d \to \mathbb{R}$: $\eta \in \partial\psi(\zeta)$ iff $\langle \eta, w - \zeta \rangle \leqslant \psi(w) - \psi(\zeta)$ for every $w \in \mathbb{R}^d$.)

Set

$$\widetilde{\Psi} : u \mapsto \int_{\mathbb{T}^d} \psi(a\nabla u)\, d\xi$$

Then consider its l.s.c. envelope

$$\Psi := \inf\left\{ \liminf_{n\to\infty} \widetilde{\Psi}(u_n) \mid u_n \to u \in L^2(\mathbb{T}^d) \text{ strongly} \right\}.$$

The gradient structure of $A = \operatorname{div}(a^*\phi(a\nabla\cdot))$

On a formal level, we have that

$$A = (a\nabla)^* \circ \partial\Psi \circ (a\nabla),$$

however, **we may avoid**

- the characterization of the domain of A or the domain of Ψ,
- singular energy spaces on weighted Riemannian manifolds of the type $W^{1,p}(\mathbb{T}^d, g_a, d\xi)$, $p \approx 1$, or $BV(\mathbb{T}^d, g_a, d\xi)$ where $g_a = (a^*a)^{-1}$,
- **non**-Hilbertian Gelfand triples $V \subset H \subset V^*$,

by employing *stochastic variational inequalities (SVI).*

About the drift part

1. a monotone (multi-valued) nonlinearity $\phi : \mathbb{R}^d \to \mathbb{R}^d$ or $\phi : \mathbb{R}^d \to 2^{\mathbb{R}^d}$ of at most linear growth.

Singular case

We assume that there exist $C \geqslant 0$, $K > 0$, such that

- $\phi = \partial\psi$, where $\psi : \mathbb{R}^d \to [0,\infty)$ and $\psi(\zeta) = \rho(|\zeta|)$ for all $\zeta \in \mathbb{R}^d$,
  - where $\rho : [0,\infty) \to [0,\infty)$, convex, continuous,
  - $\rho(0) = 0$, $\lim_{r\to\infty} \rho(r) = \infty$,
  - $\rho(r) \leqslant C(1+|r|^2)$, for every $r \geqslant 0$, (*quadratic growth condition*)
  - $\rho(2r) \leqslant K\rho(r)$, for every $r \geqslant 0$ (*doubling*, or Δ_2-*condition*)

Example

Example

For $\phi = \partial\psi$, the following examples satisfy the previous conditions for $a \equiv 1$:

- $\psi(\zeta) := \frac{1}{p}|\zeta|^p$, $p \in [1, 2]$, $\zeta \in \mathbb{R}^d$,
  - leading to the *p-Laplace* drift term $\operatorname{div}[|\nabla \cdot|^{p-2}\nabla \cdot]$,
- $\psi(\zeta) := (1 + |\zeta|)\log(1 + |\zeta|) - |\zeta|$, $\zeta \in \mathbb{R}^d$,
  - leading to a special type of *logarithmic diffusion* drift term $\operatorname{div}[\log(1 + |\nabla \cdot |)\operatorname{sgn}(\nabla \cdot)]$.

Ellipticity

1. a monotone (multi-valued) nonlinearity $\phi : \mathbb{R}^d \to \mathbb{R}^d$ or $\phi : \mathbb{R}^d \to 2^{\mathbb{R}^d}$ of at most linear growth.
2. C^1-coefficient fields $a, b : \mathbb{T}^d \to \mathbb{R}^{d \times d}$.

Assume the following *uniform ellipticity condition*: $\exists \kappa > 0$:

$$|a(\xi)\zeta|^2 \geqslant \kappa|\zeta|^2 \quad \text{for all } \zeta \in \mathbb{R}^d \text{ and all } \xi \in \mathbb{T}^d.$$

For simplicity, assume a similar condition for b.

Note.

We do neither assume that $\operatorname{div} a_i = 0$ nor that $\operatorname{div} b_i = 0$, where a_i, b_i denote the rows of a, b respectively. As a consequence, the noise coefficient operator is not skew-symmetric. ($\longrightarrow$ BDG-inequality **not** applicable **in our proof**).

Stratonovich $\rightsquigarrow$ Itô

Fix a d-dimensional Wiener process $t \mapsto \beta_t = (\beta_t^1, \dots, \beta_t^d)$ on a filtered standard probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geqslant 0}, \mathbb{P})$.

Instead of the Stratonovich SPDE with linear degenerate gradient/transport noise

$$dX_t = \operatorname{div}(a^* \phi(a \nabla X_t))\, dt + \langle b \nabla X_t, \circ d\beta_t \rangle, \quad X_0 = x \in L^2(\mathbb{T}^d),$$

we actually consider the Itô SPDE (recall the formula $f(X) \circ dX = f(X)\, dX + \frac{1}{2} d[f(X), X]$)

$$dX_t = \operatorname{div}(a^* \phi(a \nabla X_t))\, dt + \frac{1}{2} L^{\mathrm{b}} X_t\, dt + \langle b \nabla X_t, d\beta_t \rangle, \quad X_0 = x \in L^2(\mathbb{T}^d),$$

where L^{b} denotes the generator of the Dirichlet form

$$\mathcal{B}(u, v) := \int_{\mathbb{T}^d} \langle b \nabla u, b \nabla v \rangle\, d\xi.$$

On a **core** of smooth functions: $L^{\mathrm{b}} u = \operatorname{div}(b^* b \nabla u)$, $u \in C^\infty(\mathbb{T}^d)$. See e.g. [Friz, P.K./Hairer, M., **Ch. 12**, Springer (2014)] for more general transport noise with other **nonlinear drift terms**.

A weighted Riemannian manifold

Set $g_a := (a^*a)^{-1}$. Then $\mathbb{T}^d$, with the Lebesgue measure $d\xi$, is a weighted Riemannian manifold with metric g_a and with density $\rho_a := \sqrt{\det(a^*a)}$ w.r.t. the Riemannian volume.

Let $L^a u = \operatorname{div}(a^*a\nabla u)$ be the Dirichlet operator associated to the Dirichlet form

$$\mathcal{A}(u,v) := \int_{\mathbb{T}^d} \langle a\nabla u, a\nabla v\rangle \, d\xi.$$

Then $L^a = \Delta^a$ is the *Laplace-Beltrami* operator on $(\mathbb{T}^d, g_a, d\xi)$.

Curvature-dimension condition

Definition

Let $\Lambda^a := \{f \in H^1(\mathbb{T}^d) \ : \ L^a f \in H^1(\mathbb{T}^d)\}$. We say that $(\mathbb{T}^d, g_a, d\xi)$ satisfies a *Bakry-Émery curvature-dimension condition* $BE(K,\infty)$ if there exists $K \in \mathbb{R}$ such that

$$L^a|a\nabla f|^2 - 2\langle a\nabla f, a\nabla L^a f\rangle \geqslant \frac{K}{2}|a\nabla f|^2 \quad \forall f \in \Lambda^a.$$

Let $(P_t^a)_{t\geqslant 0}$ be the heat semigroup associated to $\mathcal{A}$.

Theorem (cf. [Wang, F.-Y., *Bull. Sci. Math.* (2011)])

$BE(K,\infty)$ is equivalent to

$$|a\nabla P_t^a f| \leqslant e^{-2Kt} P_t^a |a\nabla f| \quad \forall t \geqslant 0 \, \forall f \in C^1(\mathbb{T}^d).$$

Defective commutation condition

Definition

Let $R : H^1(\mathbb{T}^d) \to L^2(\mathbb{T}^d; \mathbb{R}^d)$ be a bounded linear operator such that there exists $C > 0$ with

$$\|Ru\|^2_{L^2(\mathbb{T}^d;\mathbb{R}^d)} \leqslant C\mathcal{A}(u,u) \quad \forall u \in H^1(\mathbb{T}^d).$$

We say that the *weak defective commutation property* holds for the operators $u \mapsto b\nabla u$ and L^a if there exists an operator R with the above properties such that for every $\beta > 0$,

$$b\nabla G^a_\beta u = G^a_\beta b\nabla u + G^a_\beta R G^a_\beta u \quad \forall u \in H^1(\mathbb{T}^d),$$

where $G^a_\beta := (\beta - L^a)^{-1}$, $\beta > 0$ denotes the *resolvent* of L^a.

The weak defective commuation property e.g. holds if

$$b\nabla L^a = L^a b\nabla + R$$

holds on some core of L^a, see [Shigekawa, I., *J. Funct. Anal.* (2006)]. This can be viewed a kind of *stochastic parabolicity condition*.

Compare also with the *Weitzenböck formula* $\Box^a = \Delta^a + \mathrm{R}$, where $\Box^a := -(dd^* + d^*d)$ is the *Hodge-de Rham Laplace*, and R is the curvature tensor.

Lemma

The weak defective commutation condition implies that there exists a constant $c \in \mathbb{R}$ such that for every $\beta > 0$, we have that

$$\beta \int_{\mathbb{T}^d} \langle \beta G^a_\beta b\nabla f - \beta b\nabla G^a_\beta f, b\nabla f\rangle \, d\xi \geqslant c\mathcal{A}(f,f) \quad \forall f \in H^1(\mathbb{T}^d). \tag{R}$$

Recall that $\beta(\beta G^a_\beta u - u) \to L^a u$ as $\beta \to \infty$ for $u \in D(L^a)$.

Example: rigid motions

By [Sumitomo, *Hokkaido Math. J.* (1972)], it is necessary and sufficient for the first order operators $B_i := \langle b_i, \nabla \cdot \rangle$ (where b_i are the rows of b) to commute with the Laplace-Beltrami operator L^b on a core of *smooth functions* that

$$b_i \quad \text{is a Killing vector field,}$$

meaning that, the Jakobian of b_i is skew-symmetric, i.e.

$$\partial_j b_i^k + \partial_k b_i^j = 0 \quad \forall 1 \leqslant j, k \leqslant d.$$

This automatically implies that $\operatorname{div} b_i = 0$. These vector fields generate the *group of direct affine isometries* or *rigid motions,* that is, the Lie group $SE(d)$.

In this particular case, by [Shigekawa, I., *Acta Appl. Math.* (2000)], we get the weak (~~defective~~) commutation, whenever $a = b$.

Approximation of the equation

Consider solutions $X = X^{n,\lambda,\delta,\varepsilon,m}$ to

$$dX_t = \left[\operatorname{div}(a^*\phi^\lambda(a\nabla X_t)) + \varepsilon L^a X_t + \frac{1}{2} J^a_\delta L^b J^a_\delta X_t\right] dt + \sum_{i=1}^{d} \langle b_i, \nabla J^a_\delta(\eta_m * X_t)\rangle d\beta^i_t,$$
$$X_0 = x_n \in H^1(\mathbb{T}^d),$$

where $\varepsilon > 0$ and

- $J^a_\delta := (1 - \delta L^a)^{-1} = \delta^{-1} G^a_{1/\delta}$, $\delta > 0$, denotes the resolvent of the Laplace operator,
- ϕ^λ, $\lambda > 0$, denotes the Yosida approximation of ϕ,
- $b = (b_1, \dots, b_d)^*$,
- $(\eta_m)_{m\in\mathbb{N}}$ denotes a standard mollifier on $\mathbb{T}^d$.

Lemma (Limit solutions)

For $m \to \infty$, $x \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; L^2(\mathbb{T}^d))$ there exists a unique limit solution with continuous paths, denoted by $X = X^{n,\lambda,\delta,\varepsilon}$, to the approximating equation*

$$dX_t = \left[\operatorname{div}(a^*\phi^\lambda(a\nabla X_t)) + \varepsilon L^a X_t + \frac{1}{2} J^a_\delta L^b J^a_\delta X_t\right] dt + \sum_{i=1}^d \langle b_i, \nabla J^a_\delta(X_t)\rangle d\beta^i_t,$$

$$X_0 = x_n \in H^1(\mathbb{T}^d).$$

Proof.

Fixed-point and perturbation argument, see [Gess, B./**T**, *JMPA* (2014)]. □

---

*i..e., $\mathbb{E}\left[\sup_{t\in[0,T]} \|X_t^{m,x_m} - X_t^x\|^2_{L^2}\right] \to 0$ as $m \to \infty$ for every sequence of initial conditions with $x_m \in H^1$ such that $\mathbb{E}\|x_m - x\|^2 \to 0$ and for every sequence of noise coefficients $\|B^m - B\|_{L_2(\mathbb{R}^d, L^2)} \to 0$.

Two a priori estimates

Proposition ([T, *SPDERF, in honor of Michael Röckner, Springer* (2018)])

Under the above **geometric conditions**, $X = X^{n,\lambda,\delta,\varepsilon}$ *satisfies the following two energy bounds:*

$$\operatorname*{ess\,sup}_{t\in[0,T]} \left[\mathbb{E}\|X_t\|_{L^2}^2\right] + 2\mathbb{E}\int_0^T \int_{\mathbb{T}^d} \psi^\lambda(a\nabla J_\delta X_t)\, d\xi\, dt + 2\varepsilon\mathbb{E}\int_0^T \mathcal{A}(X_t, X_t)\, dt \leqslant \mathbb{E}\|x_n\|_{L^2}^2,$$

$$\operatorname*{ess\,sup}_{t\in[0,T]} [\mathbb{E}\mathcal{A}(X_t, X_t)] + 2\varepsilon\mathbb{E}\int_0^T \|L^a X_t\|_{L^2}^2\, dt \leqslant e^{-CT}\mathcal{A}(x_n, x_n).$$

The second estimate changes slightly if $K \leqslant 0$ or $c \leqslant 0$ in the curvature-dimension condition on a or condition (R) for b respectively.

Stochastic variational inequality (SVI)

Denote $H = L^2(\mathbb{T}^d)$, $S = H^1(\mathbb{T}^d)$. Let $U := \mathbb{R}^N$. Denote by $L_2(U, H)$ the space of linear Hilbert-Schmidt operators from U to H.

Denote $B(x)\zeta := \sum_{i=1}^N \langle b_i, \nabla x\rangle \zeta^i$. Denote by Ψ the l.s.c. envelope of

$$u \mapsto \begin{cases} \int_{\mathbb{T}^d} \psi(a\nabla u)\, d\xi, & u \in H^1(\mathbb{T}^d), \\ +\infty, & u \in L^2(\mathbb{T}^d) \setminus H^1(\mathbb{T}^d). \end{cases}$$

Definition

Let $x \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$, $T > 0$. A progressively measurable map $X \in L^2([0, T] \times \Omega; H)$ is said to be an *SVI-solution* to (SPDE) if there exists a constant $C > 0$ such that

- *(Regularity)*

$$\operatorname*{ess\,sup}_{t\in[0,T]} \mathbb{E}\|X_t\|_H^2 + 2\mathbb{E}\int_0^T \Psi(X_s)\, ds \leqslant \mathbb{E}\|x\|_H^2.$$

Definition (*cont'd*)

- *(Variational inequality)* For every *admissible test-function* $Z \in L^2([0,T]\times\Omega; D(L^{\mathrm{b}}))$, that is, there are $Z_0 \in L^2(\Omega,\mathcal{F}_0,\mathbb{P};S)$, $G \in L^2([0,T]\times\Omega;H)$, $P \in L(H)$ such that G is progressively measurable, such that $P(D(L^{\mathrm{b}})) \subset D(L^{\mathrm{b}})$ and such that

$$Z_t = Z_0 + \int_0^t G_s\,ds + \frac{1}{2}\int_0^t P^* L^{\mathrm{b}} P Z_s\,ds + \int_0^t BPZ_s\,d\beta_s \quad \forall t \in [0,T],$$

we have that

$$\begin{aligned}
&\mathbb{E}\|X_t - Z_t\|_H^2 + 2\mathbb{E}\int_0^t \Psi(X_s)\,ds\\
\leqslant&\mathbb{E}\|x - Z_0\|_H^2 + 2\mathbb{E}\int_0^t \Psi(Z_s)\,ds\\
&-2\mathbb{E}\int_0^t (G_s, X_s - Z_s)_H\,ds\\
&-\mathbb{E}\int_0^t (L^{\mathrm{b}}PZ, PX - X)_H\,ds - \mathbb{E}\int_0^t (X, L^{\mathrm{b}}(Z - PZ))_H\,ds
\end{aligned}$$

for almost all $t \in [0,T]$.

The main result

Theorem (T, (2018+) https://arxiv.org/abs/1803.07005)

Suppose that the conditions from the beginning hold for ψ. Suppose that a, b are bounded and uniformly elliptic. Suppose that the curvature-dimension condition holds for a. Suppose that the weak defective commutation property holds for $b\nabla$ and L^a.[†]

Then there exists a unique adapted time-continuous SVI-solution $X \in C([0,T]; L^2(\Omega; L^2(\mathbb{T}^d)))$ to (SPDE) for every initial datum $x \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; L^2(\mathbb{T}^d))$ for every finite time-horizon $T > 0$ such that

$$\operatorname*{ess\,sup}_{t\in[0,T]} \mathbb{E}\|X_t - Y_t\|^2_{L^2(\mathbb{T}^d)} \leqslant \mathbb{E}\|x - y\|^2_{L^2(\mathbb{T}^d)}$$

for any other SVI-solution $Y \in C([0,T]; L^2(\Omega; L^2(\mathbb{T}^d)))$ starting in $y \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; L^2(\mathbb{T}^d))$.

[†] Assuming only condition (R) here is o.k., however, one then needs an additional technical condition on the domain of L^b.

Remarks on previous results

Previously ... for nonlinear, singular drift (i.e. $p < 2$) and linear gradient/transport noise of the above type:

- Idea: Doss (1976), Sussmann (1978), Brzeźniak, Capiński, Flandoli (1988), ...
  - Stratonovich-to-Itô transformation, commuting noise coefficients.
- Tubaro (1988), Kunita (*monograph* 1990), ...
  - Method of stochastic characteristics.
- [Barbu, V./Brzeźniak, Z./Hausenblas, E./Tubaro, L., *Stochastic Processes Appl.* (2013)]:
  - Smooth bounded domain $\mathcal{O} \subset \mathbb{R}^d$, Dirichlet or Neumann boundary conditions,
  - In a Gelfand triple, e.g.: $W^{1,p}(\mathcal{O}) \hookrightarrow L^2(\mathcal{O}) \hookrightarrow W^{-1,q}(\mathcal{O})$, $p > 1$, $p^{-1} + q^{-1} = 1$. Method uses convex conjugates,
  - $\langle b_i, \nabla \cdot \rangle$ and $\langle b_j, \nabla \cdot \rangle$, $i \neq j$ generate C_0-groups that are assumed to commute.

- [Ciotir, I./**T**, *J. Funct. Anal.* (2016)]:
  - Neumann boundary conditions on a convex bounded domain $\mathcal{O} \subset \mathbb{R}^d$,
  - $b_i \in C^2$, $\partial\mathcal{O} \in C^3$, commutation,
  - $p = 1$ — total variation flow with gradient noise.
- [Munteanu, I./Röckner, M., *Infin. Dimens. Anal. Quantum. Probab. Relat. Top.* (2018)]:
  - Dirichlet boundary conditions on a bounded domain $\mathcal{O} \subset \mathbb{R}^d$,
  - $b_i \in C^2$, $\partial\mathcal{O} \in C^3$, commutation,
  - $p = 1$ — total variation (TV-) flow with gradient noise.
  - Positivity and finite time extinction results.
- [Barbu, V./Brzeźniak, Z./Tubaro, L., *Appl. Math. Optim.* (2017)]:
  - L^1-theory under general assumptions (Brézis-Ekeland method — convex conjugates),
  - no uniqueness for the TV-flow and weak continuity of paths.

Ambrosio, L., Gigli, N., Savaré, G.: Bakry–Émery curvature-dimension condition and Riemannian Ricci curvature bounds.

Ann. Probab. **43**(1), 339–404 (2015)

Barbu, V., Brzeźniak, Z., Hausenblas, E., Tubaro, L.: Existence and convergence results for infinite dimensional nonlinear stochastic equations with multiplicative noise.

Stochastic Process. Appl. **123**(3), 934–951 (2013)

Barbu, V., Brzeźniak, Z., Tubaro, L.: Stochastic nonlinear parabolic equations with Stratonovich gradient noise.

Appl. Math. Optim. **188**(3), 1–17 (2017)

Barbu, V., Röckner, M.: Stochastic variational inequalities and applications to the total variation flow perturbed by linear multiplicative noise.

Arch. Ration. Mech. Anal. **209**(3), 797–834 (2013)

Ciotir, I., **T**: Nonlinear stochastic partial differential equations with singular diffusivity and gradient Stratonovich noise.

J. Funct. Anal. **271**(7), 1764–1792 (2016)

Gess, B., Röckner, M.: Stochastic variational inequalities and regularity for degenerate stochastic partial differential equations.

Trans. Amer. Math. Soc. **369**(5), 3017–3045 (2017)

Gess, B., **T**: Multi-valued, singular stochastic evolution inclusions.

J. Math. Pures Appl. **101**(6), 789–827 (2014)

Gess, B., **T**: Stability of solutions to stochastic partial differential equations.

J. Differential Equations **260**(6), 4973–5025 (2016)

Munteanu, I., Röckner, M.: The total variation flow perturbed by gradient linear multiplicative noise.

IDAQP **21**(1), 1–28 (2018)

Shigekawa, I.: Semigroup domination on a Riemannian manifold with boundary.

Acta Appl. Math. **63**(1-3), 385–410 (2000)

Shigekawa, I.: Defective intertwining property and generator domain.

J. Funct. Anal., **239**(2), 357–374 (2006)

T: Estimates for nonlinear stochastic partial differential equations with gradient noise via Dirichlet forms.

In A. Eberle et al., eds., *Stochastic Partial Differential Equations and Related Fields, In Honor of Michael Röckner, in press*, Springer Proceedings in Mathematics & Statistics (2018)

Wang, F.-Y.: Equivalent semigroup properties for the curvature-dimension condition.

Bull. Sci. Math. **135**(6-7), 803–815 (2011)

Wang, F.-Y. and Yan, L.: Gradient estimate on convex domains and applications.

Proc. Amer. Math. Soc., **141**(3), 1067–1081 (2013).

Thank you for your attention — 谢谢！