Gradient flows for the stochastic Amari neural field model

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joint work with

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Section 1

The stochastic Amari neural field model
Stochastic PDE for mean-field cortex activity

Amari-type neural field model [Amari, *Biological Cybernetics* (1977)].

\[ dU_t(x) = \left[ -\alpha U_t(x) + \int_B w(x, y)f(U_t(y)) \, dy \right] dt + \varepsilon B \, dW_t(x) \]
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The stochastic Amari neural field model

The model

Stochastic PDE for mean-field cortex activity

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\[
\frac{dU_t(x)}{dt} = \left[ -\alpha U_t(x) + \int_B w(x,y)f(U_t(y)) \, dy \right] dt + \varepsilon B dW_t(x)
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- \( w : B \times B \rightarrow \mathbb{R} \) kernel, modeling neural connectivity,
- \( f : \mathbb{R} \rightarrow (0, +\infty) \) gain function, modeling neural input,
- \( \{W_t\}_{t \geq 0} \) cylindrical Wiener process with values in \( H := L^2(B) \), modeled on \((\Omega, \mathcal{F}, \mathbb{P})\); additive noise coefficient \( B \in L(H) \).
Neuron

[OpenStax, Anatomy & Physiology (2018)]
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The stochastic Amari neural field model

The model

Neuron / neural fields

$g \sim$ voltage, $f \sim$ gain, $w \sim$ connectivity

[OpenStax, Anatomy & Physiology (2018)]

[Coombes, beim Graben, Potthast (2014)]

In: Neural Fields. Springer]
Typical gain functions $f$

Assume that $f : \mathbb{R} \rightarrow \mathbb{R}$ is globally Lipschitz. Let $F : H \rightarrow H$, $F(v)(x) := f(v(x))$, $v \in H = L^2(B)$ be the Nemytskii operator. Typically examples for this model are ($f > 0$)

$$f(s) = (1 + e^{-s})^{-1} \quad (Sigmoid)$$
Typical gain functions $f$

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\[
f(s) = (1 + e^{-s})^{-1} \quad (\text{Sigmoid}) \quad \quad f(s) = \frac{1}{2}(\tanh(s) + 1)
\]
The kernel

Let \( w : \mathcal{B} \times \mathcal{B} \to \mathbb{R} \) be measurable such that:

<table>
<thead>
<tr>
<th>Assumption 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( w(x, y) = w(y, x) ) for a.e. ( x, y \in \mathcal{B} ),</td>
</tr>
<tr>
<td>2. ( w \in L^2(\mathcal{B} \times \mathcal{B}) \cap C(\mathcal{B} \times \mathcal{B}) ),</td>
</tr>
<tr>
<td>3. ( w ) satisfies ( \sum_{i,j=1}^{n} c_i c_j w(x_i, x_j) \geq 0 )</td>
</tr>
</tbody>
</table>

for every \( n \in \mathbb{N} \), for every \( \{x_1, \ldots, x_n\} \subset \mathcal{B} \), and for every \( \{c_1, \ldots, c_n\} \subset \mathbb{R} \).
Gradient flows for the stochastic Amari neural field model

The stochastic Amari neural field model

The model

\[ w(x, y) = J(x - y) \]

Then Assumption 1 implies that the linear operator \( K \in L(H) \) defined by

\[ K g(x) := \int_{\mathcal{B}} w(x, y) g(y) \, dy, \quad g \in H, \]

is a nonnegative definite, self-adjoint Hilbert-Schmidt operator and, moreover, even of trace-class (\( \longrightarrow \text{Mercer's theorem} \)), as \( w \) is a so-called \textit{Mercer kernel} on a compact subset of \( \mathbb{R}^d \).

Assumption 2

Let \( J \in C(\mathbb{R}^d) \) such that \( w(x, y) = J(x - y) \) for \( x, y \in \mathcal{B} \).

Now, \( w \) satisfies Assumption 1 (3) e.g. if \( J \) is of the form

\[ J(x) = \int_{\mathbb{R}^d} \cos(\langle y, x \rangle) \sigma(dy), \quad x \in \mathbb{R}^d, \]

for some \textit{symmetric} probability measure \( \sigma \) on \( (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \) \( \longrightarrow \text{Bochner's theorem} \).
Examples of connectivity kernels $d = 1$

In this case, $J$ is a real-valued *characteristic function* ("Fourier transform") of a symmetric probability distribution $\sigma$.

\[
\begin{align*}
J(x) & \quad \exp \left( -\frac{x^2}{2} \right) \quad \exp(-|x|) \quad \left(1 + \frac{x^2}{2}\right)^{-1} \\
\sigma & \quad \text{Gaussian} \quad \text{Cauchy} \quad \text{Laplace}
\end{align*}
\]

\[
\begin{align*}
J(x) & \quad \frac{\sin(x)}{x} \quad (1 - x^2) \exp \left( -\frac{x^2}{2} \right) \quad (1 - |x|) \exp(-|x|) \\
\sigma & \quad \text{Uniform on } [-1, 1]
\end{align*}
\]
Examples of connectivity kernels $d = 1$

In this case, $J$ is a real-valued characteristic function ("Fourier transform") of a symmetric probability distribution $\sigma$.

| $J(x)$                           | $\exp\left(-\frac{x^2}{2}\right)$ | $\exp(-|x|)$ | $\left(1 + \frac{x^2}{2}\right)^{-1}$ |
|----------------------------------|-------------------------------------|---------------|----------------------------------------|
| $\sigma$                        | Gaussian                             | Cauchy        | Laplace                                 |

| $J(x)$                           | $\frac{\sin(x)}{x}$                | $(1 - x^2) \exp\left(-\frac{x^2}{2}\right)$ | $(1 - |x|) \exp(-|x|)$ |
|----------------------------------|-------------------------------------|---------------------------------------------|----------------------|
| $\sigma$                        | Uniform on $[-1, 1]$                | $\text{Mexican hat}$                        | $\text{Wizard hat}$  |
The stochastic PDE

Let \( \{W_t\}_{t \geq 0} \) be a cylindrical Wiener process with values in \( H = L^2(B) \) on a filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\), \( B \in L(H) \).

\[
dU_t = \left[ -\alpha U_t + KF(U_t) \right] dt + \epsilon B dW_t, \quad U_0 = u_0 \in H, \ t \in [0, T]. \tag{*}
\]

**Theorem ([Da Prato, Zabczyk, *Cambridge Univ. Press (1992)*])**

Let \( B \in L_2(H) \). Then there exists a unique mild solution to (*) with \( U \in C([0, T]; H) \), \( \mathbb{P} \)-a.s. having the form

\[
U_t = e^{-\alpha t}u_0 + \int_0^t e^{-\alpha(t-s)}KF(U_s) \, ds + \epsilon \int_0^t e^{-\alpha(t-s)} B \, dW_s \quad \text{in } H.
\]
Spatial regularity for additive noise


Assume that $\{BW_t\}_{t \geq 0}$ is of the following form

$$BW_t(x) = \sum_{i=1}^{\infty} \lambda_i v_i(x) \beta^i_t$$

with $\{\beta^i_t\}_{i \in \mathbb{N}, t \geq 0}$ independent standard Brownian motions, $v_i : \mathcal{B} \to \mathbb{R}$ Lipschitz with constants $L_i$ such that for some $\rho \in (0, 1)$

$$\sup_{x \in \mathcal{B}} \left| \sum_{i=1}^{\infty} \lambda_i^2 v_i(x)^2 \right| < \infty, \quad \sup_{x \in \mathcal{B}} \left| \sum_{i=1}^{\infty} \lambda_i^2 L_i^{2\rho} |v_i(x)|^{2(1-\rho)} \right| < \infty.$$

Then $U \in C([0, T], C(\mathcal{B}))$, whenever $U_0 \in C(\mathcal{B})$. 

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The stochastic PDE
Typical behavior in 1D

\( \mathcal{B} = [-80, 80] \), \( J \) centered Gaussian with standard deviation \( = 0.05 \),

\( f(s) = (s + 1)(1 - s)(s - 0.1) \), \( \alpha = 0.1 \), \( \epsilon = 0.5 \) with space time white noise, \( u(x, 0) = 0.8 \)
Section 2

A gradient flow formulation
Gradient flows

In fact even for $\varepsilon = 0$, it was previously not known, whether one can find a gradient structure to rewrite the PDE as

$$\partial_t u = -\nabla_X F(u), \quad u(\cdot, t) = u(t) \in \mathcal{X},$$

where $\mathcal{X}$ is a suitable function space — Hilbert, Banach, or metric space (see e.g. [Ambrosio, Gigli, Savaré, *Birkhäuser* (2006)]) — and where

$$F : \mathcal{X} \to \mathbb{R}$$

is a functional, which has often the natural interpretation of an energy, entropy or some other physical notion.

As seen in [Kuehn, Riedler, *J. Math. Neuroscience* (2014)], the naïve guess

$$F(u) := \int_B \left[ \frac{\alpha}{2} u(x)^2 - \int_B \int_0^{u(x)} f(r) w(x, y) \, dr \, dy \right] \, dx$$

in $\mathcal{X} := L^2(B)$ fails to produce the desired formulation.
Recall: \( K \) is trace-class, nonnegative definite, self-adjoint. Hence:

- The spectrum \( \sigma(K) \) is discrete with zero being its only accumulation point.
- There exists an orthonormal basis \( \{e_i\} \) of eigenvectors in \( L^2(B) \) such that the eigenvalues \( \lambda_i \in \sigma(K) \setminus \{0\} \) satisfy w.l.o.g.

\[
\lim_{i \to \infty} \lambda_i = 0.
\]

We have the orthogonal decomposition

\[
H = \text{Ker}(K) \oplus S
\]

where \( S := \text{Ker}(K)^\perp = \text{span}\{e_i\}_{i \in \mathbb{N}} \).

On \( S \) becomes a separable Hilbert space (denoted by \( H_{-1} \)) with norm

\[
\|u\|_{-1} := \|K^{-\frac{1}{2}}u\|_H \quad u \in S,
\]

where \( K^{-\frac{1}{2}} \) is the Moore-Penrose pseudo-inverse of \( K^{\frac{1}{2}} \).
Let \( \varphi : \mathbb{R} \to \mathbb{R} \) be any \textbf{primitive function} of \( f \), i.e. \( \varphi' = f \). Set
\[
\Phi(u) := \int_{\mathcal{B}} \varphi(u(x)) \, dx \quad u \in H,
\]
and let
\[
\Psi(v) := \frac{\alpha}{2} \|v\|^2_{-1}, \quad u \in S.
\]

**Lemma**

\( \Phi \) is well-defined, finite for all \( u \in H \) and continuous in \( H \). Furthermore, we have that
\[
D\Phi(u)h = (F(u), h)_H, \quad u, h \in H,
\]

where \( D\Phi(u)h \) denotes the \textbf{Gâteaux-directional derivative} of \( \Phi \) in \( u \) in direction \( h \).

---

**Gradients**

Gradient flows for the stochastic Amari neural field model

A gradient flow formulation

Change of ambient space
Furthermore, set $\Theta(u) := \Psi(u) - \Phi|_S(u)$, $u \in S$.

\begin{lemma}
For $u, h \in H_{-1}$, we have that

$$D\Theta(u)h = \alpha(u, h)_{-1} - (KF(u), h)_{-1},$$

where $D\Theta(u)h$ denotes the Gâteaux-directional derivative of $\Phi$ in $u$ in direction $h$.
\end{lemma}


However, in our situation, $F$ is not assumed monotone.
Inhibition and excitation

Remark

In the case that $K$ is nonpositive definite, we can redefine $H_{-1}$ by replacing $K$ by $-K$ in the definition. Now, by changing the sign for $\Theta$ above, we obtain a gradient by a similar procedure. We can interpret the case of nonnegative definite symmetric kernels as domination by *excitation*, while the case of nonpositive definite kernels corresponds to domination of the *inhibition* effects.
Gradient flow formulation

Let \( \{W_t\}_{t \geq 0} \) be as above. Let \( B \in L(H, H_{-1}) \). Consider the gradient flow SPDE

\[
dV_t = -D\Theta(V_t) \, dt + \varepsilon B \, dW_t, \quad V_0 = v_0 \in H_{-1}.
\]

Assumption

Assume the regularity condition

\[
B \in L_2(H, H_{-1}) \quad \text{and} \quad BK^{-1} \in L_2(H_{-1}, H).
\] \hspace{1cm} (*)
Gradient flow formulation (noise regularity)

In the simpler case that $B$ is diagonalized w.r.t. $\{e_i\}$ with eigenvalues $\{b_i\}$, i.e.

$$Be_i = b_i e_i, \quad i \in \mathbb{N},$$

we have that $K$ and $B$ commute and the second assumption in (*) can be dropped.

Clearly,

$$\{b_i^2 \lambda_i^{-1}\} \in \ell^1 \iff B \in L_2(H, H_{-1}).$$

In this case,

$$BW_t = \sum_{i=1}^{\infty} b_i e_i \beta^i_t,$$

with $\{\beta^i_t\}_{i \in \mathbb{N}}$ independent standard Brownian motions on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$.

One possibility is to set $B := K$, which corresponds to the continuum limit of a neural Langevin equation, see [Bressloff, J. Phys. A (2012)].
Gradient flows for the stochastic Amari neural field model

A gradient flow formulation

Change of ambient space

Gradient flow formulation (invariant subspace)

Theorem (Kuehn, T. (2019), invariance of the subspace $H_{-1}$)

For $B$ and $K$ satisfying (*), there exists a unique mild solution \( \{V_t\}_{t \geq 0} \) in $H_{-1}$ such that, in particular, for $v_0 \in H_{-1}$, we have that $V \in L^2([0, T]; H_{-1})$ $\mathbb{P}$-a.s. and there exist constants $C_1, C_2 > 0$ with

$$
\mathbb{E} \left[ \sup_{t \in [0, T]} \|V_t\|_{-1}^2 \right] \leq C_1 \|u_0\|_{-1}^2 + \varepsilon^2 C_1.
$$

In particular, the mild solution in $H_{-1}$ coincides with the mild solution in $H$ for initial data in $H_{-1}$.

$$
C_1 = 2 \exp \left( 2 \left( |f(0)| + \text{Lip}(f) \|K\|_{L(H)} - \alpha \right) T \right),
$$

$$
C_2 = C_1 T \left( \kappa \|B\|^2_{L_2(H,H_{-1})} + \|K^{-1}B\|^2_{L_2(H,H_{-1})} \right),
$$

with $\kappa = \kappa(C_1, T) > 1$. 

Proposition

Let $B$ and $K$ satisfy $(\ast)$. If for fixed $\omega \in \Omega$, $t \mapsto BW_t(\omega)$ is càdlàg in $H_{-1}$ and $BW(\omega) \in L^2([0, T]; H_{-1})$, we have for any initial datum $v_0 \in H_{-1}$ that the path $t \mapsto V_t(\omega)$ is weakly continuous in $H_{-1}$ and strongly right-continuous in $H_{-1}$.
Gradient flows for the stochastic Amari neural field model

A gradient flow formulation

Invariant measures

Invariant measures

Let \( \{W_t\}_{t \geq 0} \) be a cylindrical Wiener process with values in \( H \). Consider solutions to the equation

\[
dX_t^{z,\varepsilon} = (AX_t^{z,\varepsilon} + D\Phi(X_t^{z,\varepsilon})) \, dt + \varepsilon \, dW_t, \quad X_0^{z,\varepsilon} = z \in H_{-1},
\]

with \((Au, \cdot)_{-1} := -\alpha(K^{-1}u, \cdot)_H\) (which is the generator of a \( C_0 \)-semigroup \( \{S_t\}_{t \geq 0} \) on \( H_{-1} \) which is the restriction of a \( C_0 \)-semigroup \( \{S^0_t\}_{t \geq 0} \) on \( H \)).

Recall that \( D\Phi(u)h = (F(u), h)_H, \ u, h \in H \).

**Remark**

We may w.l.o.g. assume that \( t \mapsto \int_0^t S^0_{t-s} \, dW_s \in C([0, T]; H_{-1}) \) \( \mathbb{P} \)-a.s. as the semigroup \( \{S^0_t\}_{t \geq 0} \) is *analytic*. 
Invariant measures (existence)

Define the transition semigroup

\[ P^\varepsilon_t G(z) := \mathbb{E} \left[ G(X^\varepsilon_t z) \right] \quad t \geq 0, \ z \in H_{-1}, \]

where \( G : H_{-1} \to \mathbb{R} \) is bounded and measurable.

**Theorem** (by applying results from [Zabczyk, *SPDEs and Appl. II* (1989), Da Prato, Zabczyk, *Cambridge Univ. Press* (1992)])

Assume that \( B = K \). Then \( \{ P^\varepsilon_t \}_{t \geq 0} \) is strongly Markovian and symmetric with respect to its invariant measure, which exists and takes the following form

\[ \mu^\varepsilon(dz) := \frac{1}{Z^\varepsilon} \exp \left[ 2\varepsilon^{-2}\Phi(z) \right] \gamma^\varepsilon(dz), \]

where \( Z^\varepsilon := \int_{H_{-1}} \exp \left[ 2\varepsilon^{-2}\Phi(z) \right] \gamma^\varepsilon(dz) \) and \( \gamma^\varepsilon \sim N(0, \Gamma^\varepsilon) \), where \( \Gamma^\varepsilon := 2\varepsilon^2 \alpha^{-1} K \).
Invariant measures (uniqueness)

**Theorem (Compare with [Maslowski, *Stoch. Systems and Optim.* (1989)])**

Assume that $B = K$. Then $\mu_\epsilon$ is strong Feller in the restricted sense (⇒ asymptotic strong Feller) and thus unique, and the semigroup $\{P_t^\epsilon\}_{t \geq 0}$ is ergodic.

Possible applications:

- Large deviation principle / small noise asymptotics,
- Kramers’ law, (see e.g. [Berglund, *Markov Processes Related Fields* (2013)]),
- Kolmogorov operators / Fokker-Planck equations.
Final remarks

Possible extensions:

- the situation of $\mathcal{B} = \mathbb{R}^d$;
- multiplicative noise;
- locally Lipschitz gain function $f$;
- improved estimates for more specific $f$ like $sigmoid$ or $tanh$;
- indefinite kernels $K$ (however, with dominating positive or negative spectrum).
S. Amari.
Dynamics of pattern formation in lateral-inhibition type neural fields.

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Kramers’ law: validity, derivations and generalisations.

P. C. Bressloff.
Spatiotemporal dynamics of continuum neural fields.

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Stochastic neural field equations: a rigorous footing.

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Large deviations for nonlocal stochastic neural fields.

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A gradient flow formulation for the stochastic Amari neural field model.

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Strong Feller property for semilinear stochastic evolution equations and applications.

H. Wilson and J. Cowan.
A mathematical theory of the functional dynamics of cortical and thalamic nervous tissue.

J. Zabczyk.
Symmetric solutions of semilinear stochastic equations.

Thank you for your attention!