Gradient flows for the stochastic Amari neural field model

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joint work with

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Stochastic PDE for mean-field cortex activity

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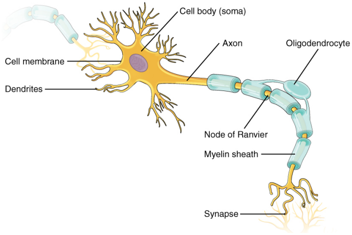
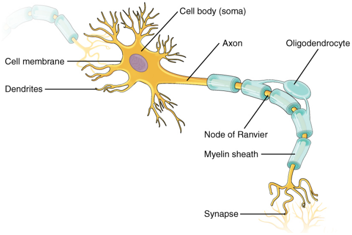
- $f : \mathbb{R} \to (0, +\infty)$ gain function, modeling neural input,
- $\{W_t\}_{t \geqslant 0}$ cylindrical Wiener process with values in $H := L^2(\mathcal{B})$, modeled on $(\Omega, \mathcal{F}, \mathbb{P})$; additive noise coefficient $B \in L(H)$.

Neuron



[OpenStax, Anatomy & Physiology (2018)]

Neuron / neural fields



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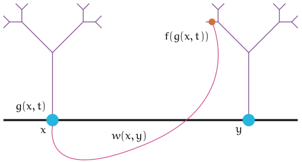


$g \sim$ voltage, $f \sim$ gain, $w \sim$ connectivity

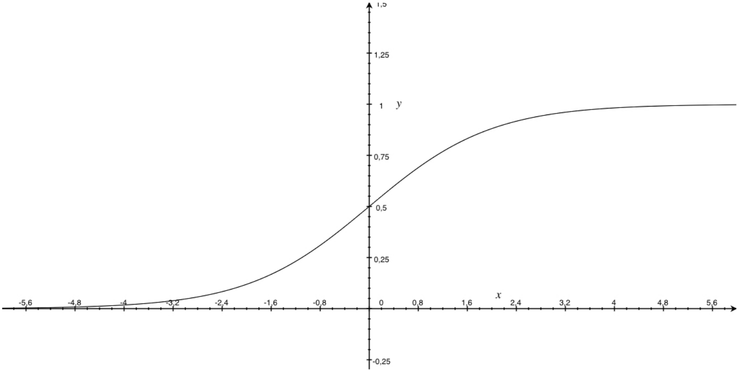
[Coombes, beim Graben, Potthast (2014)

In: Neural Fields. Springer]

Typical gain functions f

Assume that $f : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz. Let $F : H \to H$, $F(v)(x) := f(v(x))$, $v \in H = L^2(\mathcal{B})$ be the Nemytskii operator. Typically examples for this model are $(f > 0)$

$$f(s) = (1 + e^{-s})^{-1} \ (Sigmoid)$$



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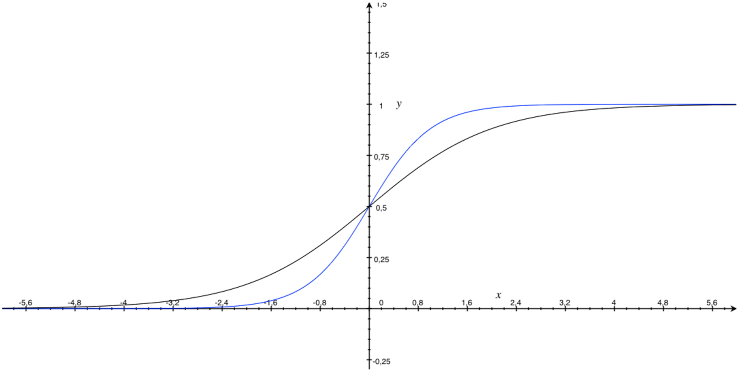
$f(s) = (1 + e^{-s})^{-1}$ (*Sigmoid*)

$f(s) = \frac{1}{2}(\tanh(s) + 1)$



The kernel

Let $w : \mathcal{B} \times \mathcal{B} \to \mathbb{R}$ be measurable such that:

Assumption 1

1. $w(x, y) = w(y, x)$ for a.e. $x, y \in \mathcal{B}$,
2. $w \in L^2(\mathcal{B} \times \mathcal{B}) \cap C(\mathcal{B} \times \mathcal{B})$,
3. w satisfies

$$\sum_{i,j=1}^{n} c_i c_j w(x_i, x_j) \geqslant 0$$

for every $n \in \mathbb{N}$, for every $\{x_1, \ldots, x_n\} \subset \mathcal{B}$, and for every $\{c_1, \ldots, c_n\} \subset \mathbb{R}$.

$w(x,y) = J(x-y)$

Then Assumption 1 implies that the linear operator $K \in L(H)$ defined by

$$Kg(x) := \int_{\mathcal{B}} w(x,y)g(y)\,dy, \quad g \in H,$$

is a nonnegative definite, self-adjoint Hilbert-Schmidt operator and, moreover, even of trace-class ($\longrightarrow$ Mercer's therorem), as w is a so-called *Mercer kernel* on a compact subset of $\mathbb{R}^d$.

Assumption 2

Let $J \in C(\mathbb{R}^d)$ such that $w(x,y) = J(x-y)$ for $x, y \in \mathcal{B}$.

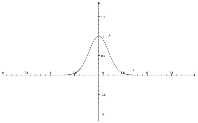
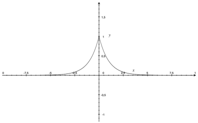
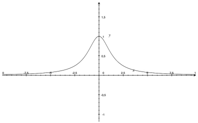
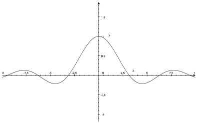
Now, w satisfies Assumption 1 (3) e.g. if J is of the form

$$J(x) = \int_{\mathbb{R}^d} \cos(\langle y, x\rangle)\,\sigma(dy), \quad x \in \mathbb{R}^d,$$

for some **symmetric** probability measure σ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \longrightarrow$ Bochner's theorem.

Examples of connectivity kernels $d = 1$

In this case, J is a real-valued *characteristic function* ("Fourier transform") of a symmetric probability distribution σ.

$J(x)$	$\exp\left(-\frac{x^2}{2}\right)$	$\exp(-\lvert x\rvert)$	$\left(1+\frac{x^2}{2}\right)^{-1}$
σ	Gaussian	Cauchy	Laplace
			
$J(x)$	$\frac{\sin(x)}{x}$	$(1-x^2)\exp\left(-\frac{x^2}{2}\right)$	$(1-\lvert x\rvert)\exp(-\lvert x\rvert)$
σ	Uniform on $[-1, 1]$		
			

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	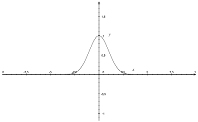	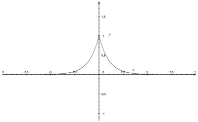	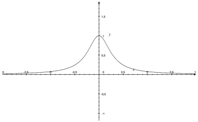
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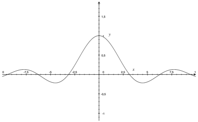
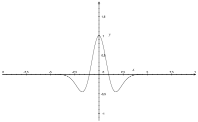
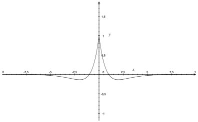
σ	Uniform on $[-1, 1]$	*Mexican hat*	*Wizard hat*
			

The stochastic PDE

Let $\{W_t\}_{t \geqslant 0}$ be a cylindrical Wiener process with values in $H = L^2(\mathcal{B})$ on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geqslant 0}, \mathbb{P})$, $B \in L(H)$.

$$dU_t = [-\alpha U_t + KF(U_t)]\, dt + \varepsilon B\, dW_t, \quad U_0 = u_0 \in H,\ t \in [0, T]. \qquad (*)$$

Theorem ([Da Prato, Zabczyk, *Cambridge Univ. Press* (1992)])

Let $B \in L_2(H)$. Then there exists a unique mild solution to $(*)$ with $U \in C([0,T]; H)$, $\mathbb{P}$-a.s. having the form

$$U_t = e^{-\alpha t} u_0 + \int_0^t e^{-\alpha(t-s)} KF(U_s)\, ds + \varepsilon \int_0^t e^{-\alpha(t-s)} B\, dW_s \quad \text{in } H.$$

Spatial regularity for additive noise

Lemma ([Kuehn, Riedler, *J. Math. Neuroscience* (2014)])

Assume that $\{BW_t\}_{t\geqslant 0}$ *is of the following form*

$$BW_t(x) = \sum_{i=1}^{\infty} \lambda_i v_i(x)\beta_t^i$$

with $\{\beta_t^i\}_{t\geqslant 0}^{i\in\mathbb{N}}$ *independent standard Brownian motions,* $v_i : \mathcal{B} \to \mathbb{R}$ *Lipschitz with constants* L_i *such that for some* $\rho \in (0,1)$

$$\sup_{x\in\mathcal{B}} \left|\sum_{i=1}^{\infty} \lambda_i^2 v_i(x)^2\right| < \infty, \quad \sup_{x\in\mathcal{B}} \left|\sum_{i=1}^{\infty} \lambda_i^2 L_i^{2\rho} |v_i(x)|^{2(1-\rho)}\right| < \infty.$$

Then $U \in C([0,T], C(\mathcal{B}))$, *whenever* $U_0 \in C(\mathcal{B})$.

Typical behavior in 1D

($\mathcal{B} = [-80, 80]$, J centered Gaussian with standard deviation $= 0.05$,
$f(s) = (s+1)(1-s)(s-0.1)$, $\alpha = 0.1$, $\varepsilon = 0.5$ with space time white noise, $u(x, 0) = 0.8$)

Section 2

A gradient flow formulation

Gradient flows

In fact even for $\varepsilon = 0$, it was previously not known, whether one can find a gradient structure to rewrite the PDE as

$$\partial_t u = -\nabla_{\mathcal{X}} \mathcal{F}(u), \quad u(\cdot, t) = u(t) \in \mathcal{X},$$

where $\mathcal{X}$ is a suitable function space — Hilbert, Banach, or metric space (see e.g. [Ambrosio, Gigli, Savaré, *Birkhäuser* (2006)]) — and where

$$\mathcal{F} : \mathcal{X} \to \mathbb{R}$$

is a functional, which has often the natural interpretation of an energy, entropy or some other physical notion.

As seen in [Kuehn, Riedler, *J. Math. Neuroscience* (2014)], the naïve guess

$$\mathcal{F}(u) := \int_{\mathcal{B}} \left[\frac{\alpha}{2} u(x)^2 - \int_{\mathcal{B}} \int_0^{u(x)} f(r) w(x, y)\, dr\, dy \right] dx$$

in $\mathcal{X} := L^2(\mathcal{B})$ fails to produce the desired formulation.

Change of ambient space

Recall: K is trace-class, nonnegative definite, self-adjoint. Hence:

- The spectrum $\sigma(K)$ is discrete with zero being its only accumulation point.
- There exists an orthonormal basis $\{e_i\}$ of eigenvectors in $L^2(\mathcal{B})$ such that the eigenvalues $\lambda_i \in \sigma(K) \setminus \{0\}$ satisfy w.l.o.g.

$$\lim_{i \to \infty} \lambda_i = 0.$$

We have the orthogonal decomposition

$$H = \mathrm{Ker}(K) \oplus S$$

where $S := \mathrm{Ker}(K)^{\perp} = \overline{\mathrm{span}\{e_i\}_{i \in \mathbb{N}}}$.

On S becomes a separable Hilbert space (denoted by H_{-1}) with norm

$$\|u\|_{-1} := \|K^{-\frac{1}{2}} u\|_H \quad u \in S,$$

where $K^{-\frac{1}{2}}$ is the Moore-Penrose pseudo-inverse of $K^{\frac{1}{2}}$.

Gradients

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be any primitive function of f, i.e. $\varphi' = f$. Set

$$\Phi(u) := \int_{\mathcal{B}} \varphi(u(x))\, dx \quad u \in H,$$

and let

$$\Psi(v) := \frac{\alpha}{2} \|v\|_{-1}^2, \quad u \in S.$$

Lemma

Φ is well-defined, finite for all $u \in H$ and continuous in H. Furthermore, we have that

$$D\Phi(u)h = (F(u), h)_H, \quad u, h \in H,$$

where $D\Phi(u)h$ denotes the Gâteaux-directional derivative of Φ in u in direction h.

Gradients

Furthermore, set $\Theta(u) := \Psi(u) - \Phi|_S(u)$, $u \in S$.

Lemma

For $u, h \in H_{-1}$, we have that

$$D\Theta(u)h = \alpha(u,h)_{-1} - (KF(u),h)_{-1},$$

where $D\Theta(u)h$ denotes the Gâteaux-directional derivative of Φ in u in direction h.

Compare also with the ideas of [Ren, Röckner, Wang, *J. Differential Equations* (2007)] and [Röckner, Wang, *J. Differential Equations* (2008)] $\longrightarrow$ generalized stochastic porous media equation.

However, in our situation, F is **not** assumed monotone.

Inhibition and excitation

Remark

In the case that K is nonpositive definite, we can redefine H_{-1} by replacing K by $-K$ in the definition. Now, by changing the sign for Θ above, we obtain a gradient by a similar procedure. We can interpret the case of nonnegative definite symmetric kernels as domination by *excitation*, while the case of nonpositive definite kernels corresponds to domination of the *inhibition* effects.

Gradient flow formulation

Let $\{W_t\}_{t \geqslant 0}$ be as above. Let $B \in L(H, H_{-1})$. Consider the gradient flow SPDE

$$dV_t = -D\Theta(V_t)\, dt + \varepsilon B\, dW_t, \quad V_0 = v_0 \in H_{-1}.$$

Assumption

Assume the regularity condition

$$B \in L_2(H, H_{-1}) \quad \text{and} \quad BK^{-1} \in L_2(H_{-1}, H). \tag{$*$}$$

Gradient flow formulation (noise regularity)

In the simpler case that B is diagonalized w.r.t. $\{e_i\}$ with eigenvalues $\{b_i\}$, i.e.

$$Be_i = b_i e_i, \quad i \in \mathbb{N},$$

we have that K and B commute and the second assumption in $(*)$ can be dropped.

Clearly, $\{b_i^2 \lambda_i^{-1}\} \in \ell^1 \iff B \in L_2(H, H_{-1}).$

In this case,

$$BW_t = \sum_{i=1}^{\infty} b_i e_i \beta_t^i,$$

with $\{\beta_t^i\}_{t\geqslant 0}^{i\in\mathbb{N}}$ independent standard Brownian motions on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$.

One possibility is to set $B := K$, which corresponds to the continuum limit of a neural Langevin equation, see [Bressloff, *J. Phys. A* (2012)].

Gradient flow formulation (invariant subspace)

Theorem (Kuehn, T. (2019), invariance of the subspace H_{-1})

For B and K satisfying $(*)$, there exists a unique mild solution $\{V_t\}_{t\geqslant 0}$ in H_{-1} such that, in particular, for $v_0 \in H_{-1}$, we have that $V \in L^2([0,T];H_{-1})$ $\mathbb{P}$-a.s. and there exist constants $C_1, C_2 > 0$ with

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|V_t\|_{-1}^2\right] \leqslant C_1\|u_0\|_{-1}^2 + \varepsilon^2 C_1,$$

In particular, the mild solution in H_{-1} coincides with the mild solution in H for initial data in H_{-1}.

$$C_1 = 2\exp\left(2\left[(|f(0)| + \mathrm{Lip}(f))\|K\|_{L(H)} - \alpha\right]T\right),$$
$$C_2 = C_1 T\left(\kappa\|B\|^2_{L_2(H,H_{-1})} + \|K^{-1}B\|^2_{L_2(H,H_{-1})}\right),$$

with $\kappa = \kappa(C_1, T) > 1$.

Pathwise regularity

Proposition

Let B and K satisfy $(*)$. If for fixed $\omega \in \Omega$, $t \mapsto BW_t(\omega)$ is càdlàg in H_{-1} and $BW(\omega) \in L^2([0,T]; H_{-1})$, we have for any intial datum $v_0 \in H_{-1}$ that the path $t \mapsto V_t(\omega)$ is weakly continuous in H_{-1} and strongly right-continuous in H_{-1}.

Invariant measures

Let $\{W_t\}_{t\geqslant 0}$ be a cylindrical Wiener process with values in H. Consider solutions to the equation

$$dX_t^{z,\varepsilon} = (AX_t^{z,\varepsilon} + D\Phi(X_t^{z,\varepsilon}))\,dt + \varepsilon\,dW_t, \quad X_0^{z,\varepsilon} = z \in H_{-1},$$

with $(Au,\cdot)_{-1} := -\alpha(K^{-1}u,\cdot)_H$ (which is the generator of a C_0-semigroup $\{S_t\}_{t\geqslant 0}$ on H_{-1} which is the restriction of a C_0-semigroup $\{S_t^0\}_{t\geqslant 0}$ on H).

Recall that $D\Phi(u)h = (F(u),h)_H,\ u,h \in H$.

Remark

We may w.l.o.g. assume that $t \mapsto \int_0^t S_{t-s}^0\,dW_s \in C([0,T];H_{-1})$ $\mathbb{P}$-a.s. as the semigroup $\{S_t^0\}_{t\geqslant 0}$ is *analytic*.

Invariant measures (existence)

Define the *transition semigroup*

$$P_t^{\varepsilon} G(z) := \mathbb{E}\left[G(X_t^{z,\varepsilon})\right] \quad t \geqslant 0,\ z \in H_{-1},$$

where $G : H_{-1} \to \mathbb{R}$ is bounded and measurable.

Theorem (by applying results from [Zabczyk, *SPDEs and Appl. II* (1989), Da Prato, Zabczyk, *Cambridge Univ. Press* (1992)])

Assume that $B = K$. Then $\{P_t^{\varepsilon}\}_{t \geqslant 0}$ is strongly Markovian and symmetric with respect to its invariant measure, which exists and takes the following form

$$\mu_{\varepsilon}(dz) := \frac{1}{Z_{\varepsilon}} \exp\left[2\varepsilon^{-2}\Phi(z)\right]\ \gamma_{\varepsilon}(dz),$$

where $Z_{\varepsilon} := \int_{H_{-1}} \exp\left[2\varepsilon^{-2}\Phi(z)\right]\ \gamma_{\varepsilon}(dz)$ and $\gamma_{\varepsilon} \sim N(0, \Gamma_{\varepsilon})$, where $\Gamma_{\varepsilon} := 2\varepsilon^2\alpha^{-1}K$.

Invariant measures (uniqueness)

Theorem (Compare with [Maslowski, *Stoch. Systems and Optim.* (1989)])

Assume that $B = K$. Then μ_ε is strong Feller in the restricted sense ($\Longrightarrow$ asymptotic strong Feller) and thus unique, and the semigroup $\{P_t^\varepsilon\}_{t\geqslant 0}$ is ergodic.

Possible applications:

- Large deviation principle / small noise asymptotics,
- Kramers' law, (see e.g. [Berglund, *Markov Processes Related Fields* (2013)]),
- Kolmogorov operators / Fokker-Planck equations.

Final remarks

Possible extensions:

- the situation of $\mathcal{B} = \mathbb{R}^d$;
- multiplicative noise;
- locally Lipschitz gain function f;
- improved estimates for more specific f like *sigmoid* or *tanh;*
- *indefinite* kernels K (however, with dominating positive or negative spectrum).

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Thank you for your attention!