Gradient flows for the stochastic Amari neural field model

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joint work with

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https://arxiv.org/abs/1807.02575

9th International Conference on Stochastic Analysis and its Applications
Bielefeld University

September 3, 2018
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Section 1

The stochastic Amari neural field model
The stochastic Amari neural field model

The model

Stochastic PDE for mean-field cortex activity

Amari-type neural field model [Amari, Biological Cybernetics (1977)].

\[
    dU_t(x) = \left[ -\alpha U_t(x) + \int_B w(x, y) f(U_t(y)) \, dy \right] dt + \varepsilon B \, dW_t(x)
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- \( f : \mathbb{R} \to (0, +\infty) \) gain function, modeling neural input,
- \( \{W_t\}_{t \geq 0} \) cylindrical Wiener process with values in \( H := L^2(B) \), modeled on \( (\Omega, \mathcal{F}, \mathbb{P}) \); additive noise coefficient \( B \in L(H) \).
Gradient flows for the stochastic Amari neural field model

The stochastic Amari neural field model

The model

Neuron

[OpenStax, Anatomy & Physiology (2018)]
Neuron / neural fields

$g \sim$ voltage, $f \sim$ gain, $w \sim$ connectivity

[OpenStax, Anatomy & Physiology (2018)]

[Coombes, beim Graben, Potthast (2014)]

In: Neural Fields. Springer]
Gradient flows for the stochastic Amari neural field model

Typical gain functions $f$

Assume that $F : H \rightarrow H$, $F(v)(x) := f(v(x))$, $v \in H = L^2(B)$ is a Nemytskii operator.

$$f(s) = \frac{1}{1 + e^{-s}} \quad (Sigmoid)$$
Typical gain functions \( f \)

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\[
f(s) = (1 + e^{-s})^{-1} \quad (\text{Sigmoid}) \\
f(s) = \frac{1}{2} (\tanh(s) + 1)
\]
The kernel

Let $w : \mathcal{B} \times \mathcal{B} \to \mathbb{R}$ be measurable such that:

**Assumption 1**

1. $w(x, y) = w(y, x)$ for a.e. $x, y \in \mathcal{B}$,
2. $w \in L^2(\mathcal{B} \times \mathcal{B}) \cap C(\mathcal{B} \times \mathcal{B})$,
3. $w$ satisfies
   \[
   \sum_{i,j=1}^{n} c_i c_j w(x_i, x_j) \geq 0
   \]
   for every $n \in \mathbb{N}$, for every $\{x_1, \ldots, x_n\} \subset \mathcal{B}$, and for every $\{c_1, \ldots, c_n\} \subset \mathbb{R}$. 
Gradient flows for the stochastic Amari neural field model

The stochastic Amari neural field model

The model

\[ w(x, y) = J(x - y) \]

Then Assumption 1 implies that the linear operator \( K \in L(H) \) defined by

\[ Kg(x) := \int_B w(x, y)g(y) \, dy, \quad g \in H, \]

is a nonnegative definite, self-adjoint Hilbert-Schmidt operator and, moreover, even of trace-class (\( \rightarrow \) Mercer’s theorem), as \( w \) is a so-called Mercer kernel on a compact subset of \( \mathbb{R}^d \).

**Assumption 2**

Let \( J \in C(\mathbb{R}^d) \) such that \( w(x, y) = J(x - y) \) for \( x, y \in B \).

Now, \( w \) satisfies Assumption 1 (3) e.g. if \( J \) is of the form

\[ J(x) = \int_{\mathbb{R}^d} \cos(\langle y, x \rangle) \sigma(dy), \quad x \in \mathbb{R}^d, \]

for some symmetric probability measure \( \sigma \) on \( (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \) \( \rightarrow \) Bochner’s theorem.
Examples $d = 1$ (characteristic functions)

In this case, $J$ is a real-valued characteristic function of a symmetric probability distribution.

- **Gaussian**
- **Cauchy**
- **Laplace**

- Uniform (on $[-1, 1]$)
- Maxwell–Boltzmann (“Mexican hat” wavelet)
Gradient flows for the stochastic Amari neural field model

The stochastic PDE

Let \( \{W_t\}_{t \geq 0} \) be a cylindrical Wiener process with values in \( H = L^2(B) \) on a filtered probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}), B \in L(H) \).

\[
dU_t = [-\alpha U_t + KF(U_t)] dt + \epsilon B dW_t, \quad U_0 = u_0 \in H, \ t \in [0, T]. \quad (\ast)
\]


Let \( B \in L_2(H) \). Then there exists a unique mild solution to (\ast) with \( U \in C([0, T]; H), \mathbb{P}\)-a.s. having the form

\[
U_t = e^{-\alpha t} u_0 + \int_0^t e^{-\alpha (t-s)} KF(U_s) \, ds + \epsilon \int_0^t e^{-\alpha (t-s)} B \, dW_s \quad \text{in } H.
\]
Section 2

A gradient flow formulation
Gradient flows

In fact even for $\varepsilon = 0$, it was not known, whether one can find a gradient structure to rewrite the PDE as

$$\partial_t u = -\nabla_X F(u), \quad u(\cdot, t) = u(t) \in \mathcal{X},$$

where $\mathcal{X}$ is a suitable function space — Hilbert, Banach, or metric space (see e.g. [Ambrosio, Gigli, Savaré, Birkhäuser (2006)]) — and where

$$\mathcal{F} : \mathcal{X} \to \mathbb{R}$$

is a functional, which has often the natural interpretation of an energy, entropy or some other physical notion.

It was already seen in [Kuehn, Riedler, J. Math. Neuroscience (2014)] that the naïve guess for $\mathcal{F}$ in $\mathcal{X} := L^2(B)$ fails to produce the desired formulation.
Change of ambient space

Recall: $K$ is trace-class, nonnegative definite, self-adjoint. Hence:

- The spectrum $\sigma(K)$ is discrete with zero being its only accumulation point.
- There exists an orthonormal basis $\{e_i\}$ of eigenvectors in $L^2(\mathcal{B})$ such that the eigenvalues $\lambda_i \in \sigma(K) \setminus \{0\}$ satisfy w.l.o.g.

$$\lim_{i\to\infty} \lambda_i = 0.$$ 

We have the orthogonal decomposition

$$H = \text{Ker}(K) \oplus S$$

where $S := \text{Ker}(K)^\perp = \text{span}\{e_i\}_{i\in\mathbb{N}}$.

On $S$ becomes a separable Hilbert space (denoted by $H_{-1}$) with norm

$$\|u\|_{-1} := \|K^{-\frac{1}{2}} u\|_H \quad u \in S,$$

where $K^{-\frac{1}{2}}$ is the Moore-Penrose pseudo-inverse of $K^{\frac{1}{2}}$. 


Gradients

Let \( \varphi : \mathbb{R} \rightarrow \mathbb{R} \) be any \textit{primitive function} of \( f \). Set

\[
\Phi(u) := \int_B \varphi(u(x)) \, dx \quad u \in H,
\]

and let

\[
\Psi(v) := \frac{\alpha}{2} \|v\|_1^2, \quad u \in S.
\]

\textbf{Lemma}

\( \Phi \) is well-defined, finite for all \( u \in H \) and continuous in \( H \). Furthermore, we have that

\[
D \Phi(u)h = (F(u), h)_H, \quad u, h \in H,
\]

where \( D \Phi(u)h \) denotes the \textit{Gâteaux-directional derivative} of \( \Phi \) in \( u \) in direction \( h \).
Furthermore, set $\Theta(u) := \Psi(u) - \Phi|_S(u)$, $u \in S$.

**Lemma**

For $u, h \in H_{-1}$, we have that

$$D\Theta(u)h = \alpha(u, h)_{-1} - (KF(u), h)_{-1},$$

where $D\Theta(u)h$ denotes the Gâteaux-directional derivative of $\Phi$ in $u$ in direction $h$.


However, in our situation, $F$ is not assumed monotone.
Inhibition and excitation

Remark

In the case that $K$ is nonpositive definite, we can redefine $H_{-1}$ by replacing $K$ by $-K$ in the definition. Now, by changing the sign for $\Theta$ above, we obtain a gradient by a similar procedure. We can interpret the case of nonnegative definite symmetric kernels as domination by *excitation*, while the case of nonpositive definite kernels corresponds to domination of the *inhibition* effects.
Gradient flows for the stochastic Amari neural field model

A gradient flow formulation

Let \( \{W_t\}_{t \geq 0} \) be as above. Let \( B \in L(H, H_{-1}) \). Consider the gradient flow SPDE

\[
dV_t = -D\Theta(V_t) \, dt + \epsilon B \, dW_t, \quad V_0 = v_0 \in H_{-1}.
\]

We shall assume below that \( B \in L_2(H, H_{-1}) \). Under the assumption that \( B \) is diagonalized w.r.t. \( \{e_i\} \) with eigenvalues \( \{b_i\} \), i.e.

\[
Be_i = b_i e_i, \quad i \in \mathbb{N},
\]

it holds that

**Lemma**

\[
\{b_i^2 \lambda_i^{-1}\} \in \ell^1 \quad \iff \quad B \in L_2(H, H_{-1}).
\]

One possibility is to set \( B := K \), which corresponds to the continuum limit of a neural Langevin equation, see [Bressloff, *J. Phys. A* (2012)].
Gradient flows for the stochastic Amari neural field model

A gradient flow formulation

Change of ambient space

Gradient flow formulation (invariant subspace)

Theorem (Kuehn, T. (2018+), invariance of the subspace $H_{-1}$)

For $B \in L^2(H.H_{-1})$, we have that there exists a unique mild solution $\{V_t\}_{t \geq 0}$ in $H_{-1}$ such that, in particular, for $v_0 \in H_{-1}$, we have that $V \in L^2([0, T]; H_{-1})$ $\mathbb{P}$-a.s. and

$$
\mathbb{E} \left[ \sup_{t \in [0, T]} \|V_t\|_{-1}^2 \right] \leq C\|u_0\|_{-1}^2 + \varepsilon^2 C,
$$

In particular, the mild solution in $H_{-1}$ coincides with the mild solution in $H$ for initial data in $H_{-1}$.

In our work, we also study strong and weak solutions to the above SPDE in $H_{-1}$ via:

- Transformation of the additive noise,
- Dissipativity of the drift.
Let \( \{W_t\}_{t \geq 0} \) be a cylindrical Wiener process with values in \( H \). Consider solutions to the equation

\[
dX_t^{z, \varepsilon} = (AX_t^{z, \varepsilon} + D\Phi(X_t^{z, \varepsilon})) \, dt + \varepsilon \, dW_t, \quad X_0^{z, \varepsilon} = z \in H_{-1},
\]

with \( (Au, \cdot)_{-1} := -\alpha(K^{-1}u, \cdot)_H \) (which is the generator of a \( C_0 \)-semigroup \( \{S_t\}_{t \geq 0} \) on \( H_{-1} \) which is the restriction of a \( C_0 \)-semigroup \( \{S_0^t\}_{t \geq 0} \) on \( H \)).

Recall that \( D\Phi(u)h = (F(u), h)_H, \ u, h \in H \).

**Remark**

We may w.l.o.g. assume that \( t \mapsto \int_0^t S_{t-s}^0 \, dW_s \in C([0, T]; H_{-1}) \) \( \mathbb{P} \)-a.s. as the semigroup \( \{S_t^0\}_{t \geq 0} \) is *analytic*. 
Invariant measures (existence)

Define the transition semigroup

$$P_t^\varepsilon G(z) := \mathbb{E} \left[ G(X_t^{\varepsilon, z}) \right] \quad t \geq 0, \ z \in H_{-1},$$

where $G : H_{-1} \to \mathbb{R}$ is bounded and measurable.

**Theorem** (by applying results from [Zabczyk, *SPDEs and Appl. II* (1989), Da Prato, Zabczyk, *Cambridge Univ. Press* (1992)])

Assume that $t \mapsto \int_0^t S_{t-s}^0 dW_s \in C([0, T]; H_{-1})$ $\mathbb{P}$-a.s. Then $\{P_t^\varepsilon\}_{t \geq 0}$ is strongly Markovian and symmetric with respect to its invariant measure, which exists and takes the following form

$$\mu_\varepsilon(dz) := \frac{1}{Z_\varepsilon} \exp \left[ 2\varepsilon^{-2} \Phi(z) \right] \gamma_\varepsilon(dz),$$

where $Z_\varepsilon := \int_{H_{-1}} \exp \left[ 2\varepsilon^{-2} \Phi(z) \right] \gamma_\varepsilon(dz)$ and $\gamma_\varepsilon \sim N(0, \Gamma_\varepsilon)$, where $\Gamma_\varepsilon := 2\varepsilon^2 \alpha^{-1} K$. 
Invariant measures (uniqueness)

**Theorem (Compare with [Maslowski, *Stoch. Systems and Optim.* (1989)])**

Assume that \( t \mapsto \int_0^t S_{t-s}^0 \, dW_s \in C([0, T]; H_{-1}) \mathbb{P}\text{-a.s.} \). Then \( \mu_\varepsilon \) is **strong Feller in the restricted sense** (asymptotic strong Feller) and thus unique, and the semigroup \( \{P_\varepsilon^t\}_{t \geq 0} \) is **ergodic**.

Possible applications:

- Large deviation principle / small noise asymptotics,
- Kramers’ law, (see e.g. [Berglund, *Markov Processes Related Fields* (2013)]),
- Kolmogorov operators / Fokker-Planck equations.
Final remarks

Possible extensions:

- Situations where $K$ and $B$ do not commute,
- the situation of $B = \mathbb{R}^d$,
- multiplicative noise,
- indefinite kernels $K$ (however, with dominating positive or negative spectrum).
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Thank you for your attention!