A new take on ergodicity of the stochastic 2D Navier-Stokes equations

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1 Motivation

Consider the stochastic incompressible 2D Navier-Stokes equations (NSE) with additive Wiener noise on a bounded domain $\mathcal{O}\subset\mathbb{R}^2$, viscosity $\nu>0$,

$$dX_t=[\nu\Delta X_t+(X_t\cdot\nabla)X_t]\,dt+B\,dW_t,\qquad t>0,$$

$$\nabla\cdot X_t=0,\qquad t\geq 0,$$

$$X_0=x\in L^2_{\mathrm{sol}}(\mathcal{O};\mathbb{R}^2)=:H,$$

with no-slip boundary condition $X_t=0$ on $\partial\mathcal{O}$.

By the spectral Galerkin method, if $B\in\mathrm{HS}(H)$ and if $\{W_t\}_{t\geq 0}$ is a cylidrical Wiener noise modelled on H, there exists a unique Markovian strong solution

$X^x\in L^2_W([0,T]\times\Omega;H^1_{\mathrm{sol},0}(\mathcal{O};\mathbb{R}^2))$.

Locally monotone drift SPDEs

The 2D NSE is an example of a *locally monotone drift SPDE*.

$$dX_t = A(X_t)\,dt + B\,dW_t, \qquad t \geq 0.$$

Let $V \subset H \equiv H^* \subset V^*$ be a Gelfand triple.

V separable, reflexive Banach space, H separable Hilbert space.

$A\colon V \to V^*$ is called *monotone* if there exists $K \in \mathbb{R}$,

$$\langle A(u) - A(v), u - v \rangle \leq K \|u - v\|_H^2, \qquad u, v \in V.$$

If $H = V = \mathbb{R}$, $f\colon \mathbb{R} \to \mathbb{R}$ is monotone **iff** $x \mapsto f(x) - Kx$ is non-increasing.

A is called *locally monotone* in V if there exists $\rho\colon V \to \mathbb{R}$, locally bounded and measurable, and $K \in \mathbb{R}$, such that

$$\langle A(u) - A(v), u - v \rangle \leq (K + \rho(u)) \|u - v\|_H^2, \qquad u, v \in H.$$

Further examples

Further examples of locally monotone drift SPDEs

- Stochastic power-law fluid equations
- Stochastic Allen-Cahn equation
- Stochastic Burgers equation

Non-monotone perturbations of monotone drift SPDEs

- Stochastic p-Laplace equation
- Stochastic porous medium-type equations

(not covered by our ergodicity result)

Stochastic power law fluids

Further examples of locally monotone drift SPDEs

- Stochastic power-law fluid equations on $\mathcal{O}\subset\mathbb{R}^d$, $p\in(1,\infty)$,

$$dX_t = [\nabla\cdot S(\boldsymbol{e}(X_t)) + (X_t\cdot\nabla)X_t]\,dt + B\,dW_t, \qquad t>0,$$

$$\nabla\cdot X_t = 0, \qquad t\geq 0,$$

$$X_0 = x\in L^2_{\mathrm{sol}}(\mathcal{O};\mathbb{R}^d),$$

where

$$\boldsymbol{e}(u)_{i,j} := \frac{1}{2}(\partial_i u_j + \partial_j u_i),$$

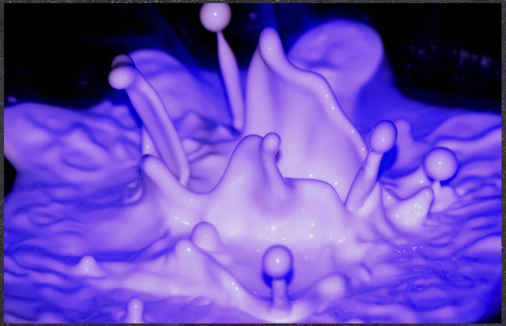
and

$$S(z) := 2\nu(1+|z|)^{p-2}z.$$

Further examples of locally monotone drift SPDEs

- Stochastic power-law fluid equations on $\mathcal{O} \subset \mathbb{R}^d$, $p \in (1, \infty)$, $d = 2, 3$.

$p > 2$	$p < 2$	$p = 2$
dilatant, or *shear-thickening*	*pseudoplastic*, or *shear-thinning*	*Newtonian*, Navier-Stokes equations
oobleck	hair gel, blood, whipped cream	Examples: water, glycerol, ethanol
(mixture of water and corn-starch)	(polymeric molecules)	viscous stress $\propto$ local strain rate



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Stochastic Allen-Cahn equation

Further examples of locally monotone drift SPDEs

- Stochastic Allen-Cahn equation on $\mathcal{O} \subset \mathbb{R}^d$, $d = 1, 2, 3$.

$$dX_t = [\nu \Delta X_t + g(X_t)]\, dt + B\, dW_t, \qquad t > 0,$$

$$X_0 = x \in L^2(\mathcal{O}).$$

Typically, $g(z) = z - z^3$.

- Stochastic Burgers equation on $\mathcal{O} \subset \mathbb{R}^d$, $d = 1, 2, 3$.

$$dX_t = [\nu \Delta X_t + \langle \boldsymbol{f}(X_t), \nabla X_t \rangle]\, dt + B\, dW_t, \qquad t > 0,$$

$$X_0 = x \in L^2(\mathcal{O}).$$

1D stochastic Burgers, $d = 1$, $\boldsymbol{f}(z) = z$.

Main question

Existence and uniqueness of solutions to the examples for any finite time horizon $T > 0$ have been discussed in Liu, Röckner [LR10].

What about $T \to \infty$?

Replacing the noise $B\,dW$ by a deterministic forcing $f \in L^2_{\mathrm{sol}}$, the 2D NSE with no-slip boundary condition is known to have **exponential convergence** to the stationary solution [Tem01], when **the viscosity is large enough** relative to constants depending only on the **domain**, the **first eigenvalue** of the Stokes operator, and the L^2_{sol}-norm of the **deterministic forcing** f.

Without forcing, the solution has **exponential decay** to zero.

What about the stochastic case?

2 Invariant measures

Set $P_tF(x) := \mathbb{E}[F(X_t^x)]$, $t \geq 0$, $F \in \mathcal{B}_b(H)$, $x \in H$ and define P_t^* by duality

$$\langle P_t^*\mu, F\rangle = \langle P_tF, \mu\rangle, \quad t \geq 0,\ \mu \in \mathcal{M}_1(H),\ F \in C_b(H).$$

Definition 1. *A probability measure $\mu \in \mathcal{M}_1(H)$ is said to be* invariant *for $\{P_t\}$ if $P_t^*\mu = \mu$ for every $t \geq 0$.*

$\{P_t\}_{t\geq 0}$ is called Feller *if $P_t(C_b(H)) \subset C_b(H)$ for every $t \geq 0$.*

$\{P_t\}$ is called weak* mean ergodic if for some invariant measure μ,

$$\frac{1}{T}\int_0^T P_t^*\nu\, dt \rightharpoonup \mu, \quad \text{as} \quad T \to \infty$$

for every $\nu \in \mathcal{M}_1(H)$ which is equivalent to the uniqueness of μ.

Proposition 2. *(Krylov-Bogoliubov).*

If for a Feller semigroup $\{P_t\}$ *and some* $x \in H$, $t_n \nearrow \infty$, $\mu \in \mathcal{M}_1(H)$,

$$\frac{1}{t_n}\int_0^{t_n} \mathrm{Law}(X_s^x)\,ds \rightharpoonup \mu \quad \text{as} \quad n \to \infty$$

then μ *is an invariant measure for* $\{P_t\}$.

Remark 3. This method can be used to prove that the stochastic 2D NSE with additive Gaussian forcing admits an invariant measure by standard a priori estimates.

Strong Feller property

Definition 4. *A Feller semigroup $\{P_t\}$ is said to have the* strong Feller property *if*

$$P_t(\mathcal{B}_b(H)) \subset C_b(H) \quad \text{for every} \quad t \geq 0.$$

A Feller semigroup $\{P_t\}$ is said to be *irreducible* if for every $t > 0$ and for every $x \in H$ and for every non-empty open set $O \subset H$,

$$P_t \mathbb{1}_O(x) > 0.$$

Theorem 5. *If $\{P_t\}$ is strong Feller and irreducible, it admits a unique invariant measure μ such that for every $x \in H$*

$$||P_t^* \delta_x - \mu||_{\mathrm{TV}} \to 0 \quad \text{as} \quad t \to \infty.$$

Timeline (some highlights)

- In 1995, [Flandoli, Maslowski] proved the strong Feller property and irreducibility of the stochastic 2D NSE with non-degenerate Gaussian noise.
- In 2001–2002, the (exponential) ergodicity of the stochastic 2D NSE with either non-degenerate forcing or large viscosity ν was proved by Bricmont, Kupiainen, and Lefevere [BKL02]; E, Mattingly, and Sinai [EMS01]; Kuksin and Shirikyan [KS01]; Mattingly [Mat02]; ...
- In 2002, the general well-posedness theory for the stochastic 2D NSE was discussed by Menaldi and Sritharan [MS02].
- In 2006, Hairer and Mattingly [HM06] provided a minimal non-degeneracy condition of the noise for the asymptotic strong Feller property and weak irreducibility.
- **And many others ...**

  Recently, there has been a lot of progess for multiplicative noise, Lévy noise, pure jump noise, and coupling techniques.

Hairer and Mattingly [HM06], [HM08] introduced the *asymptotic strong Feller property*, which admits the following **sufficient condition**.

Proposition 6. *Let* $t_n \nearrow \infty$ *and* $\delta_n \searrow 0$, *and* $C: [0, \infty) \to \mathbb{R}$ *nondecreasing. Then,* $\{P_t\}$ *is* asymptotically strong Feller *if for all* $\varphi \in \mathrm{Lip}_b(H)$,

$$|\nabla P_{t_n} \varphi(x)| \leq C(\|x\|)(\|\varphi\|_\infty + \delta_n \|\nabla \varphi\|_\infty) \quad x \in H, \, n \in \mathbb{N}.$$

Definition 7. $\{P_t\}$ *is called* weakly irreducible *if for every* $x, y \in H$ *there exists* $z \in H$ *such that for any* $z \in O \subset H$ *open, there exists* $s, t > 0$ *with*

$$P_s \mathbb{1}_O(x) > 0 \quad \text{and} \quad P_t \mathbb{1}_O(y) > 0.$$

Theorem 8. *Weak irreducibility* + *Asymptotic strong Feller* $\Rightarrow$ *Unique ergodicity.*

Vorticity formulation

For the stochastic 2D NSE on the torus $\mathbb{T}^2 = [-\pi, \pi]^2$ in vorticity formulation

$$dv_t = [\nu \Delta v_t + u_t \cdot \nabla v_t] dt + \tilde{B} dW_t$$

where $v = \partial_2 u^1 - \partial_1 u^2$ is the vorticity, and u the velocity (which can be recovered by the Biot-Savart formula), one considers additive noise of the form

$\tilde{B} dW_t = \sum_{\mathbf{k}} \tilde{B} h_{\mathbf{k}} d\beta_t^{\mathbf{k}}$, where $\{\beta_t^{\mathbf{k}}\}_{t \geq 0}$, $\mathbf{k} \in \mathbb{Z}^2 \setminus \{(0,0)\}$ are i.i.d. standard Brownian motions in $\mathbb{R}$, and $\{h_{\mathbf{k}}\}$ is the standard orthonormal basis (ONB) for $L^2(\mathbb{T}^2)$

$$h_{\mathbf{k}}(\xi) = \begin{cases} \sin(\mathbf{k} \cdot \xi), & \mathbf{k} \in \mathbb{Z}_+^2 \\ \cos(\mathbf{k} \cdot \xi), & \mathbf{k} \in \mathbb{Z}_-^2 \end{cases}, \quad \xi \in \mathbb{T}^2.$$

Minimal non-degeneracy condition

Let $\mathcal{Z}_0$ be a finite dimensional subset of $\mathbb{Z}^2 \setminus \{(0,0)\}$ with

$\tilde{B}h_{\mathbf{k}} = 0$ when $\mathbf{k} \notin \mathcal{Z}_0$ and $\tilde{B}h_{\mathbf{k}} \neq 0$ when $\mathbf{k} \in \mathcal{Z}_0$.

The main result of [HM06] is the following minimal non-degeneracy condition.

Theorem 9. *Assume that $\mathcal{Z}_0 \subset \mathbb{Z}^2 \setminus \{(0,0)\}$ is symmetric and finite dimensional such that*

i. *There exist at least two elements in $\mathcal{Z}_0$ with different Euclidean norms.*

ii. *Integer linear combinations of $\mathcal{Z}_0$ generate $\mathbb{Z}^2$.*

Then the stochastic 2D NSE in vorticity formulation admits a unique invariant measure.

Example 10. $\mathcal{Z}_0 = \{(1,0), (-1,0), (1,1), (-1,-1)\}$.

Mild degeneracy

Denote by P_n the Galerkin projection on the first n Fourier modes. If the noise coefficient B is *mildly degenerate*, that is, $B \in \mathrm{HS}(H)$ and for every $\nu > 0$, there exists $N = N(\nu, \|B\|_{\mathrm{HS}(H)})$ such that, if

$$\mathrm{Rg}(B) \supset P_n(H),$$

for some $n \geq N$, then the stochastic 2D NSE admits a unique invariant measure, and the *Foias-Prodi estimate* holds in the stochastic case for some $C = C(x, y, B, \mathcal{O}, \nu) > 0$, and $\delta = \delta(B, \mathcal{O}, \nu) > 0$ such that

$$\mathbb{E}[\|X_t^x - X_t^y\|_H^2] \leq Ce^{-\delta t}$$

see e.g. [GMR17]. Thus, we get exponential mixing. The estimates rely on the properties of the **exponential martingale**.

The e-property

Definition 11. *A Feller semigroup* $\{P_t\}$ *is said to satisfy the* e-property *if for every* $\varphi \in \mathrm{Lip}_b(H)$*, for every* $x \in H$*, for every* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that*

$$|P_t\varphi(x) - P_t\varphi(y)| < \varepsilon$$

for every $t \geq 0$ and for every $y \in H$ with $\|x - y\| < \delta$.

This type of uniform equicontinuity for bounded Lipschitz functions could be viewed a **coupling condition at infinity**.

It has been conjectured by Szarek and Worm [SW12] that

> "It seems that all known examples of Markov processes with the asymptotic strong Feller property satisfy the e-property as well."

Jaroszewska constructed a counter-example in 2013 in an unpublished preprint.

The "lower bound technique"

Originally developed by Lasota and Szarek [LS06], the "lower bound technique" can be described as follows in the work of Komorowski, Peszat and Szarek [KPS10].

Theorem 12. *Assume that $\{P_t\}$ is Feller and has the e-property. Assume that there exists $z \in H$ such that for every bounded set $J \subset H$ and every $\delta > 0$*

$$\inf_{x \in J} \liminf_{T \to \infty} \frac{1}{T} \int_0^T P_t \mathbb{1}_{B(z,\delta)}(x) dt > 0.$$

Suppose further that for every $\varepsilon > 0$ and every $x \in H$ there exists a bounded Borel set $K \subset H$ such that

$$\liminf_{T \to \infty} \frac{1}{T} \int_0^T P_t \mathbb{1}_K(x) dt > 1 - \varepsilon.$$

Then there exists a unique invariant probability measure μ for $\{P_t\}$, and $\{P_t\}$ is weak* mean ergodic.

Consider

$$(1) \quad dX_t = A(X_t)\,dt + dL_t, \qquad t > 0, \qquad X_0 = x_0,$$

where the **additive Lévy noise** is given by the Itô increment of

$$L_t = BW_t + \int_{\{\|z\| \le 1\}} G(z) \tilde{N}(\{t\}, dz)$$

where $\{BW_t\}$ is as before (i.e., $\{W_t\}$ is a **cylindrical Wiener process** on H and $B \in \mathrm{HS}(H)$), and $\tilde{N}((0,t] \times U) = N((0,t] \times U) - t\pi(U)$ is an independent **compensated Poisson random measure** with σ-finite intensity measure π on $\mathcal{B}(Z)$, where Z is a Banach space, such that $\pi(\{\|z\| > 1\}) < \infty$ (large jumps).

Let $G\colon Z \to H$ be strongly measurable with $\int_{\{\|z\| \le 1\}} \|G(z)\|_H^2 \, \pi(dz) < \infty$.

As we consider just small jumps, $\{L_t\}$ is a **martingale**.

Hypotheses on the drift

The drift $A\colon V \to V^*$ is *hemicontinuous* (i.e., weakly continuous along rays) and for $\alpha \geq 2$, $\delta_1 > 0$, $u \in V$,

$$2\langle A(u), u\rangle \leq \delta_1 \|u\|_V^\alpha \quad \text{(coercivity)}$$

and for $\delta_2 > 0$, $C_2 \geq 0$, $u, v \in V$,

$$2\langle A(u) - A(v), u - v\rangle \leq (-\delta_2 + \rho(u))\|u - v\|_H^2, \quad \text{(local monotonicity)}$$

where for $\beta \geq 0$

$$0 \leq \rho(u) \leq C_2 \|u\|_V^\alpha \|u\|_H^\beta \quad u \in V.$$

Moreover, for $K > 0$, $u \in V$,

$$\|A(u)\|_{V^*}^{\frac{\alpha}{\alpha-1}} \leq K(1 + \|u\|_V^\alpha)(1 + \|u\|_H^\beta) \quad \text{(boundedness)}.$$

Existence and uniqueness of solutions

Liu and Röckner (for Wiener noise) [LR10] and Brzezniak, Liu, and Zhu (for Lévy noise) [BLZ14] proved the following.

Theorem 13. *Under the previous hypotheses, for every initial datum*

$$x_0 \in L^{\beta+2}(\Omega, \mathcal{F}_0, \mathbb{P}; H)$$

there exists a unique strong adapted càdlàg solution $\{X_t\}$ *with*

$$X \in L^{\alpha}([0,T]; V) \cap L^2([0,T]; H) \quad \mathbb{P}\text{-}a.s.$$

such that every progressively measurable V*-valued version of* X *satisfies (1)* $\mathbb{P}$*-a.s.*

Note that our assumptions are intentionally not the most general ones, and exclude e.g. the stochastic p-Laplace equation, $p \neq 2$, or time-dependent drift.

Non-standard hypothesis on the drift

Assume also that there exist $\delta_4 > 0$ and $C_4 \in \mathbb{R}$ such that for all $u \in V$,

$$2\langle A(u), u\rangle \leq C_4 - \delta_4 \|A(u)\|_{V^*} \quad \text{(cone condition)}.$$

Remark 14. This condition is satisfied for the 2D NSE and the power law fluid equations. Unfortunately, it is quite restrictive for *semilinear equations* with drift $A = A_0 + F$ because it forces the nonlinear perturbation F of the dissipative principal term A_0 to have at most quadratic growth.

For this reason, our result does not cover stochastic Allen-Cahn equations with cubic nonlinearity.

Non-standard hypothesis on the noise

Assume that $\int_{\{\|z\|\leq 1\}} \|G(z)\|_H^{\beta+2} \pi(dz) < \infty$, and that $\{L_t\}$ has the cylindrical representation

$$L_t = \sum_{k=1}^{\infty} \sqrt{\lambda_k} e_k l_t^k, \quad t \geq 0,$$

where $e_k \in V$, $k \in \mathbb{N}$ form an ONB of H, $\{\lambda_k\} \in \ell^1$ and $\{l_t^k\}$ are i.i.d. symmetric Lévy processes in $\mathbb{R}$ with finite second moments.

Moreover, assume that for some $\|e_k\|_V \leq \sqrt{\sigma_k}$ with $\{\lambda_k \sigma_k\} \in \ell^1$ it holds that

$$\|\sum_{k=1}^{\infty} \sqrt{\lambda_k} e_k l_t^k\|_V^2 \leq \sum_{k=1}^{\infty} \lambda_k \sigma_k |l_t^k|^2.$$

If V is a separable Hilbert space, it is sufficient to assume that $\{\|e_k\|_V^{-1} e_k\}$ is an ONB of V.

Remark 15. If $G \equiv 0$, and V is a separable Hilbert space, the previous assumptions are equivalent to assuming that $B \in \mathrm{HS}(H, V)$.

The symmetry and additional regularity of the Lévy process is needed to prove the *small ball property* in V.

Theorem 16. *(Barrera, T., arXiv:2412.01381) Under the previous hypotheses, the semigroup associated to (1) is Markovian and Feller, and satisfies the e-property and is weak* mean ergodic. The unique invariant probability measure μ on $(H, \mathcal{B}(H))$ admits finite $(\alpha+\beta)$-moments in H. If $\beta = 0$, μ admits α-moments in V.*

The existence and uniqueness of μ is proved by the e-property and the lower-bound technique.

Theorem 17. *Under the hypotheses on the noise, the stochastic 2D NSE with additive Lévy noise admits a unique invariant measure with finite fourth moments.*

In particular, we cover the case of $O = \mathbb{T}^2$ and e.g. $\mathcal{Z}_0 = \{(1,1)\}$ (2 Fourier modes), $\mathcal{Z}_0 = \{(1,0)\}$ (1 Fourier mode) or $\mathcal{Z}_0 = \emptyset$ (the deterministic case), etc.

We point out that ergodicity and exponential mixing of passive scalars advected the velocity of the stochastic 2D NSE for less than 4 modes has been proved recently by Kooperman and Rowan [CR24] for initial data in $H^5_{\mathrm{sol},0}(\mathbb{T}^2, \mathbb{R}^2)$.

Our result is true for all $\nu > 0$ and for initial data in $L^2_{\mathrm{sol}}(\mathcal{O}, \mathbb{R}^2)$, however, we do not obtain a convergence rate. We get weak convergence of the time-averages of the law. Mixing for all $\nu > 0$ and degenerate noise is an open problem.

Here, $\rho(u) = \frac{64}{\nu^3}\|u\|^4_{L^4_{\mathrm{sol}}}$, $\delta_2 = \nu c_0^2$, where c_0 is the inverse Poincaré constant of $\mathcal{O}$.

Ergodicity of the power law fluid equation

We also get ergodicity for the shear-thickening stochastic incompressible power law fluid equations with additive Lévy noise (the “oobleck case”).

Theorem 18. *Under the hypotheses on the noise, the stochastic power law fluid equation for* $p \geq \frac{2+d}{2} \vee 2$ *with additive Lévy noise admits a unique invariant measure with finite* $\frac{2p}{2p-d}$*th moments.*

4 Sketch of the proof

We first need to verify the e-property.

By a Galerkin approximation, Itô's lemma, and the Burkholder-Davis-Gundy inequality, we obtain the following a priori esimate for some constant $C>0$, not depending on $T>0$, and small enough $\gamma>0$,

$$\frac{1}{2}\mathbb{E}\left[\sup_{0\le t\le T}\|X_t\|_H^{\beta+2}\right]+\delta_1\frac{\beta+2}{4}\mathbb{E}\int_0^T e^{\gamma(t-T)}\|X_t\|_V^{\alpha}\|X_t\|_H^{\beta}dt$$

$$\le\mathbb{E}[\|x_0\|^{\beta+2}e^{-\gamma T}]+\frac{C}{\gamma}(1-e^{\gamma(t-T)})\le\mathbb{E}[\|x_0\|^{\beta+2}]+\frac{C}{\gamma}.$$

For the 2D NSE, these types of estimates are well-known. In particular, applied for $\beta=0$, we obtain the existence of at least one invariant measure by the Krylov-Bogoliubov theorem.

Recall:

Definition 19. *A Feller semigroup* $\{P_t\}$ *is said to satisfy the* e-property *if for every* $\varphi \in \mathrm{Lip}_b(H)$, *for every* $x \in H$, *for every* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that*

$$|P_t\varphi(x) - P_t\varphi(y)| < \varepsilon$$

for every $t \geq 0$ and for every $y \in H$ with $\|x - y\| < \delta$.

Naïve proof.

$$|P_t\varphi(x) - P_t\varphi(y)| \leq \|\varphi\|^2_{\mathrm{Lip}} \mathbb{E}[\|X_t^x - X_t^y\|_H^2]$$

Note that if A is monotone, we get that

$$\mathbb{E}[\|X_t^x - X_t^y\|_H^2] = \|x - y\|_H^2 + 2\mathbb{E}\int_0^t \langle A(X_s^x) - A(X_s^y), X_s^x - X_s^y \rangle ds \leq \|x - y\|_H^2.$$

Naïve idea: Gronwall lemma

Note that because we have additive noise, for $0 < \gamma < \delta_2$, by local monotonicity,

$$\|X_T^x - X_T^y\|_H^2$$

$$= \|x - y\|_H^2 e^{-\gamma T} + C_2 \int_0^T e^{\gamma(t-T)} \|X_t^x - X_t^y\|_H^2 ((\gamma - \delta_2) + \|X_t^x\|_V^\alpha \|X_t^x\|_H^\beta) dt$$

Hence by the Gronwall lemma,

$$\mathbb{E}\|X_T^x - X_T^y\|_H^2 + (\delta_2 - \gamma)\mathbb{E}\int_0^T e^{\gamma(t-T)} \|X_t^x - X_t^y\|_H^2 dt$$

$$\leq \|x - y\|_H^2 \, \mathbb{E}\left[\exp\left(C_2 \int_0^T e^{\gamma(t-T)} \|X_t^x\|_V^\alpha \|X_t^x\|_H^\beta dt\right)\right]$$

However, we do not have any control over the mean exponential.

Stochastic Gronwall lemma [AV24]

Theorem 20. *Let* $s \geq 0$, *let* τ *be a stopping time with values in* $[s, \infty)$. *Let* X, Y, $f\colon [s,\tau) \times \Omega \to [0,\infty)$ *be progressively measurable processes such that a.s.* X *has increasing and continuous paths, a.s.* $Y \in L^1_{\mathrm{loc}}([s,\tau))$, *and a.s.* $f \in L^1(s,\tau)$. *Supposet that there exist constants* $\eta \geq 0$ *and* $C \geq 1$ *such that for all stopping times* $s \leq \lambda \leq \Lambda \leq \tau$

$$\mathbb{E}[X(\Lambda)] + \mathbb{E}\int_\lambda^\Lambda Y(t)\,dt \leq C(\mathbb{E}[X(\lambda)] + \eta) + \mathbb{E}\Big[(X(\Lambda)+\eta)\int_\lambda^\Lambda f(t)\,dt\Big],$$

Then $X(\tau) + \int_s^\tau Y(t)\,dt$ *is finite a.s. and for every* $\varepsilon > 0$, $R > 0$,

$$\mathbb{P}\Big(X(\tau) + \int_s^\tau Y(t)\,dt \geq \varepsilon\Big) \leq \frac{4C}{\varepsilon} e^{4CR}(\mathbb{E}[X(0)] + \eta) + \mathbb{P}\Big(\int_s^\tau f(t)\,dt \geq R\Big).$$

Note. This enables us to dispense with *exponential martingales.*

Application of the stochastic Gronwall lemma

Recall for $0 < \gamma < \delta_2$,

$$\sup_{0\le t\le T} \|X_t^x - X_t^y\|_H^2$$

$$\le \|x-y\|_H^2 + C_2 \sup_{0\le t\le T} \|X_t^x - X_t^y\|_H^2 \int_0^T e^{\gamma(t-T)} \|X_t^x\|_V^\alpha \|X_t^x\|_H^\beta dt.$$

Now, by the stochastic Gronwall lemma [AV24], and the Markov inequality,

$$\mathbb{P}\left(\sup_{0\le t\le T} \|X_t^x - X_t^y\|_H^2 + (\delta_2 - \gamma) \int_0^T \|X_t^x - X_t^y\|_H^2 dt \ge \frac{\varepsilon^2}{9\|\varphi\|_{\mathrm{Lip}}} \right)$$

$$\le \frac{36}{\varepsilon^2} \|\varphi\|_{\mathrm{Lip}}^2 e^{4R} \|x-y\|_H^2 + \frac{1}{R} \mathbb{E}\left[C_2 \int_0^T e^{\gamma(t-T)} \|X_t^x\|_V^\alpha \|X_t^x\|_H^\beta dt \right].$$

Proof cont'd

Recall the a priori estimate

$$\delta_1 \frac{\beta+2}{4} \mathbb{E} \int_0^T e^{\gamma(t-T)} \|X_t^x\|_V^\alpha \|X_t^x\|_H^\beta dt \le \|x\|^{\beta+2} + \frac{C}{\gamma}.$$

Hence, for fixed x and for any $T \ge 0$, by choosing suitable $R > 0$ and $\delta > 0$, recalling that $\|x - y\|_H \le \delta$,

$$|P_t\varphi(x) - P_t\varphi(y)| \le \frac{\varepsilon}{3} + 2\|\varphi\|_\infty \mathbb{P}\left(\sup_{0 \le t \le T} \|X_t^x - X_t^y\|_H^2 \ge \frac{\varepsilon^2}{9\|\varphi\|_{\text{Lip}}} \right)$$

$$\le \frac{\varepsilon}{3} + 2\|\varphi\|_\infty \left(\frac{36}{\varepsilon^2} \|\varphi\|_{\text{Lip}}^2 e^{4R} \|x - y\|_H^2 + \frac{1}{R} \mathbb{E}\left[C_2 \int_0^T e^{\gamma(t-T)} \|X_t^x\|_V^\alpha \|X_t^x\|_H^\beta dt \right] \right)$$

$$\le \frac{\varepsilon}{3} + 2\|\varphi\|_\infty \left(\frac{36}{\varepsilon^2} \|\varphi\|_{\text{Lip}}^2 e^{4R} \delta^2 + \frac{4C_2}{\delta_1(\beta+2)R} \left(\|x\|_H^{\beta+2} + \frac{C}{\gamma} \right) \right) \le \varepsilon.$$

Stochastic stability

Consider the deterministic counterpart to (1),

$$du_t^x = A(u_t^x)\,dt, \quad t > 0, \quad u_0^x = x,$$

and note that by coercivity, for every $R > 0$,

$$\lim_{t\to\infty} \sup_{\|x\|_H^\alpha \le R} \|u_t^x\|_H = 0.$$

Let us prove that for any $T > 0$, for any $\varepsilon > 0$, and any $K \subset H$ bounded, we have

$$\mathbb{P}(\|X_T^x - u_T^x\|_H^2 < \varepsilon) > 0,$$

uniformly for $x \in K$.

The "lower bound"

To prove this **stochastic stability** for finite times with positive probability, we use a pathwise argument to estimate $\|Y_t^x - u_t^x\|_H^2$, where $Y_t^x := X_t^x - L_t^x$ is the solution to the random PDE

$$dY_t = A(Y_t + L_t)\,dt, \quad t > 0, \quad Y_0 = x.$$

To control the error terms with positive probability, we need the *small ball property* of $\{L_t\}$ in V, that is,

$$\mathbb{P}\left(\sup_{0\le t\le T} \|L_t\|_V < \delta\right) > 0.$$

which we can prove for symmetric Lévy processes in V. For the pathwise argument, we also need the non-standard assumption that there exist $\delta_4 > 0$ and $C_4 \in \mathbb{R}$ such that for all $u \in V$,

$$2\langle A(u), u\rangle \le C_4 - \delta_4 \|A(u)\|_{V^*}.$$

The "lower bound" cont'd

We obtain

$$\mathbb{P}\left(\sup_{0 \le t \le T} \|X_t^x - u_t^x\|_H^2 < \varepsilon \right)$$

$$\geq \mathbb{P}\left(\sup_{0 \le t \le T} \|Y_t^x - u_t^x\|_H^2 < \frac{\varepsilon}{4}, \sup_{0 \le t \le T} \|L_t\|_H^2 < \frac{\varepsilon}{4} \right) > 0.$$

However, in the estimates for $\sup_{0 \le t \le T} \|Y_t^x - u_t^x\|_H^2$, we in fact need the stronger requirement

$$\mathbb{P}\left(\sup_{0 \le t \le T} \|L_t\|_V^2 < \delta \right) > 0,$$

and an application of the stochastic Gronwall lemma.

The “lower bound” cont’d

Now, for every $\delta > 0$, and every $z \in K$, there exists $\gamma_1 > 0$, and $T_0 > 0$ such that

$$P_T \mathbb{1}_{B_\delta(0)}(z) = \mathbb{P}(\|X^z_{T_0}\|_H \leq \delta) \geq \mathbb{P}\left(\|X^z_{T_0} - u^z_{T_0}\|_H \leq \frac{\delta}{2} \right) \geq \gamma_1 > 0.$$

By a well-know trick used for instance by Es-Sarhir and von Renesse [EvR12], we may use the Markov property of the semigroup (shifting by T_0) to obtain

$$\liminf_{T\to\infty} \frac{1}{T} \int_0^T P_t \mathbb{1}_{B_\delta(0)}(x)\, dt \geq \liminf_{T\to\infty} \gamma_1 \frac{1}{T} \int_0^T P_t \mathbb{1}_K(x)\, dt > 0.$$

Here, we have also used that the coercivity implies for every $\varepsilon > 0$ and every bounded set J, there exists a bounded set K with

$$\inf_{x\in J} \liminf_{T\to\infty} \frac{1}{T} \int_0^T P_t \mathbb{1}_K(x)\, dt > 1 - \varepsilon.$$

Final remarks

Thus, the conditions of Komorowski, Peszat and Szarek [KPS10] can be verified.

This proves our main result. The moment estimates follow from our a priori estimate and the fact that $V \subset H$ is a continuous linear embedding.

Theorem 21. *(Barrera, T., arXiv:2412.01381) Under the previous hypotheses, the semigroup associated to (1) is Markovian and Feller, and satisfies the e-property and is weak* mean ergodic. The unique invariant probability measure μ on $(H, \mathcal{B}(H))$ admits finite $(\alpha+\beta)$-moments in H. If $\beta = 0$, μ admits α-moments in V.*

Remark 22.

- Due to our method, our proof is restricted to locally monotone equations, where ρ depends **only on one variable** and not on both (*fully locally monotone*).
- The V**-regularity** and **symmetry** of the noise are technical assumptions.

Wrap-up and open problems

- We have proved the **ergodicity** of locally monotone drift SPDEs with Wiener noise plus independent symmetric Poisson noise with **spatial regularity** in the more regular space V without using *exponential martingales*.
- This extends the known results for **stochastic 2D NSE** for all $\nu > 0$ by a **new minimal condition** on the Fourier modes of the noise and by the **Poisson part**. Ergodicity for *mildly degenerate Lévy noise* is well-known.
- Other examples are the **stochastic power law fluid equations** for $p > 2$, the **stochastic heat equation**, and the **stochastic 1D Burgers** equation.
- Semilinear SPDEs like the **stochastic Allen-Cahn equation**, and *fully locally monotone* SPDEs like the **stochastic Cahn-Hilliard equation** are not covered by our results, nor is the **stochastic p-Laplace equation**.
- **Mixing** and quantification of mixing times remain an open question in this general situation.

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Thank you for your attention!