Singular limits for stochastic equations*

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Contents

Section 1

Singular limits for stochastic equations

Stochastic Allen-Cahn equation

Consider the 2D stochastic Allen-Cahn (AC) equation with additive Wiener noise and periodic boundary conditions:

$$\partial_t u = \Delta u + F(u) + Q\,\partial_t W, \quad u(0) \in L^2(\mathbb{T}^2).$$

Typically, $F(u) = u - u^3$, having a double-well potential $v \mapsto \frac{1}{4}\int_{\mathbb{T}^2} \left(|v|^2 - 1\right)^2 \, dx$.

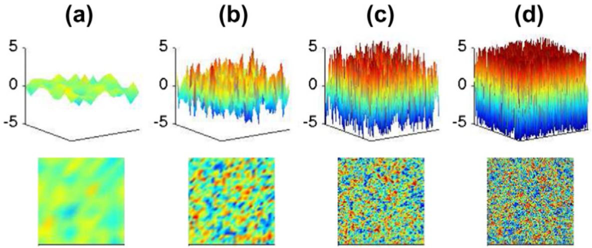
Also, for a logarithmic potential, we consider the nonlinearity

$$F(u) = \frac{\theta}{2}\log\left(\frac{1-u}{1+u}\right) + \theta_0 u, \quad -1 < u < 1, \quad \theta_0 > \theta > 0.$$

Here, $t \mapsto W(t)$ denotes a cylindrical Wiener process on L^2 and $Q : L^2 \to L^2$ a linear operator that is regularizing the spatial variable. Thus, $\partial_t W$ is space-time white noise.

Stochastic Allen-Cahn equation

The scalar solution u to the stochastic AC equation models the state of a binary alloy, where the local minima of the double-well $x \mapsto \frac{1}{4}\left(|x|^2 - 1\right)^2$ can be identified as local equilibria of concentration of the two phases ± 1. The logarithmic potential is relevant for phase field models in thermodynamics.



The 2D stochastic Allen-Cahn equation is solved numerical over a finite time interval for an increasing number of grid points: (a)–(d). The final fields are shown from side and top angles. The image is an extract of [Ryser, Nigam, Tupper (2012)].

Pattern formation

Both the deterministic and the stochastic AC equation are known for pattern formation and coarsening effects.

(a) $t = 0$ (b) $t = 500$ (c) $t = 5000$ (d) $t = 50,000$

The patterned dynamics of Allen–Cahn model solved with the simple Euler algorithm starting a random initial condition: (a–d) four typical patterns at t = 0, t = 500, t = 5000, and t = 50, 000. The image is an extract of [Zhang, Hu, Liu (2020)].

Renormalization

For $Q = \mathrm{Id}$, that is, in the case of additive space-time white noise, it is well-known that the stochastic AC equation has to be renormalized, see [Da Prato, Debussche (2003)], and see [Berglund (2022)] for an overview.

What is the reason for that?

Issue. There is no canonical way to define the product of two distributions.

More on that later.

Solution. “Renormalize” the equation.

This process can be viewed a local reparametrization leading to an implicit dependence of the solution to the specific choice of smooth approximations.

Renormalization in 1D

Generally, it is well-known that (even) in the 1D case, different classes of approximations of rough driving noises lead to any of a one-parameter family of theories of limiting stochastic integrals with different properties (where the Itô and Stratonovich stochastic integrals are most known), see [Kupferman, Pavliotis, Stuart (2004)].

> *"In practice this choice is informed either by phenomenological considerations or by a desire for the integral to satisfy a given mathematical property."*

[Bruned, Chandra, Chevyrev, Hairer (2021)]

Stochastic singular limits

Let us take an asymptotic view on renormalization. For $\varepsilon > 0$, in a Hilbert space H, let

$$\partial_t u_\varepsilon = A_\varepsilon u_\varepsilon + F_\varepsilon(u_\varepsilon) + Q_\varepsilon \, \partial_t W, \quad u_\varepsilon(0) \in H, \tag{1}$$

- $A_\varepsilon : D(A_\varepsilon) \subset H \to H$, linear (pseudo-) differential operator,
- F_ε lower-order nonlinearity (here, usually cubic),
- $t \mapsto W(t)$ standard cylindrical Wiener process on H,
- $Q_\varepsilon : H \to H$ compact linear operator, noise coefficient.

The covariance operator $Q_\varepsilon^* Q_\varepsilon$ encodes all the information about correlations and regularity of the noise.

Stochastic singular limits

We shall consider the following singular limit $\varepsilon \searrow 0$ for equation (1). Assume $Q_\varepsilon = \sigma_\varepsilon \widehat{Q}_\varepsilon$, $\sigma_\varepsilon > 0$ such that:

- $\sigma_\varepsilon \to 0$, vanishing noise intensity,
- $\widehat{Q}_\varepsilon \to \mathrm{Id}$ strongly, increasing roughness of the noise,
- $A_\varepsilon \to A$ in some suitable sense (for instance strong resolvent convergence),
- $F_\varepsilon \to F$ in some strong sense,
- $u_\varepsilon(0) \to u_0$ strongly.

Aim. Find asymptotics for σ_ε and $\widehat{Q}_\varepsilon$ such that the limit u of the u_ε unexpectedly satisfies

$$\partial_t u = Au + G(u), \quad u(0) = u_0, \tag{2}$$

with $G \neq F$. This idea is inspired by [Hairer, Ryser, Weber (2012)].

The main idea

We need to determine σ_ε such that with high probability

$$u_\varepsilon \approx u + Z_\varepsilon,$$

where Z_ε is the solution to (being a stochastic convolution)

$$dZ_\varepsilon = A_\varepsilon Z_\varepsilon\, dt + \sigma_\varepsilon \widehat{Q}_\varepsilon\, dW, \quad Z_\varepsilon(0) = 0, \tag{3}$$

and u solves the deterministic PDE

$$\partial_t u = Au + G(u), \quad u(0) = u_0. \tag{2}$$

Stochastic AC equation

With cubic F, we have that $G(u) = F(u) - 3C_0 u$, where $C_0 > 0$ is determined by the limit of the square Z_ε^2 in the weak space H^{-1}.

Section 2

Setup and main result

Family of Gelfand triples

Assumption

Suppose that we have for every $\varepsilon > 0$ a Gelfand-triple together with an additional Banach space X such that

$$V_\varepsilon \subset X \subset H \simeq H' \subset V_\varepsilon'$$

for a separable and reflexive Banach space V_ε with topological dual V_ε' such that all embeddings are continuous and dense.

Using the standard transformation, we define

$$v_\varepsilon := u_\varepsilon - Z_\varepsilon,$$

in order to obtain

$$\partial_t v_\varepsilon = A_\varepsilon v_\varepsilon + F_\varepsilon(v_\varepsilon + Z_\varepsilon), \quad v_\varepsilon(0) = u_\varepsilon(0). \tag{4}$$

Existence of solutions

Assumption

We assume that weak solutions to (2) exist in a Gelfand triple

$$V \subset X \subset H \simeq H' \subset V'$$

with dense and continuous embeddings such that

$$u \in L^2([0,T],V) \cap C^0([0,T],X) \cap H^1([0,T],V').$$

Moreover, we assume that weak solutions to (4) exist with

$$v_\varepsilon \in L^2([0,T],V_\varepsilon) \cap C^0([0,T],X) \cap H^1([0,T],V_\varepsilon').$$

Suppose moreover that the error $\varphi_\varepsilon = u - v_\varepsilon$ is well-defined with

$$\varphi_\varepsilon \in L^2([0,T],V_\varepsilon) \cap H^1([0,T],V_\varepsilon').$$

Remark

Varying energy spaces

There is a hidden relation between V and V_ε, as we pose assumptions at $\varphi_\varepsilon = u - v_\varepsilon$. This is related to the convergence $A_\varepsilon \to A$ which we shall encode in the a priori estimate and the asymptotics of a residual term Res_ε. We shall not assume any abstract relation of V_ε and V as the spaces are usually canonically attached to A_ε and A respectively.

Residual

Define

$$\begin{aligned}\mathrm{Res}_\varepsilon(u)(t) :=& \partial_t u(t) - A_\varepsilon u(t) - F_\varepsilon(u(t) + Z_\varepsilon(t)) \\ =& (A - A_\varepsilon)u(t) + F(u(t) + Z_\varepsilon(t)) - F_\varepsilon(u(t) + Z_\varepsilon(t)) \\ &+ G(u(t)) - F(u(t) + Z_\varepsilon(t))\end{aligned}$$

For the approximation result consider the error with u, v_ε:

$$\varphi_\varepsilon = v_\varepsilon - u = u_\varepsilon - Z_\varepsilon - u$$

which (using the residual) solves

$$\partial_t \varphi_\varepsilon = A_\varepsilon \varphi_\varepsilon + F_\varepsilon(\varphi_\varepsilon + u + Z_\varepsilon) - F_\varepsilon(u + Z_\varepsilon) + \mathrm{Res}_\varepsilon(u).$$

Crucial asymptotic estimate

Assumption

Consider the spaces X and V_ε and suppose $F_\varepsilon : X \to V_\varepsilon'$ and $A_\varepsilon : V_\varepsilon \to V_\varepsilon'$. Assume that there exist constants $C \geqslant 0$, $\delta > 0$, and $0 \leqslant c_\varepsilon \to 0$ for $\varepsilon \to 0$, such that:

$$\langle A_\varepsilon \varphi, \varphi \rangle + \langle F_\varepsilon(\varphi + \psi) - F_\varepsilon(\psi), \varphi \rangle \leqslant -\delta \|\varphi\|_{V_\varepsilon}^2 + C\|\varphi\|_H^2$$

for every choice of $\varphi \in V_\varepsilon$ and $\psi \in X$ with $c_\varepsilon \|\psi\|_X \leqslant 1$.

Main convergence result

Theorem (Blömker, T, to appear in Stochastics & Dynamics (2023+), arXiv:2204.09545)

Under our assumptions, let u_ε, u be any two solutions to (1), (2), respectively, and let Z_ε be the solution to (3). Then, for all $T > 0$ there is a constant $K > 0$ such that for $\varepsilon > 0$ sufficiently small we have that for all $t \in [0, T]$ and all $\gamma > 0$:

$$\mathbb{P}\left(\sup_{[0,T]} \|u_\varepsilon - u - Z_\varepsilon\|_H^2 > K\gamma\right)$$

$$\leqslant \mathbb{P}\left(\sup_{[0,T]} \|Z_\varepsilon\|_X > (2c_\varepsilon)^{-1}\right) + \mathbb{P}\left(\|u(0) - u_\varepsilon(0)\|_H^2 > \gamma\right)$$

$$+ \mathbb{P}\left(\int_0^T \|\operatorname{Res}_\varepsilon(u)\|_{V_\varepsilon'}^2 \, dt > \gamma\right).$$

Note that for $c_\varepsilon > 0$ the condition $\varepsilon > 0$ being sufficiently small in the previous theorem can be quantified by $\sup_{[0,T]} \|u\|_X \leqslant (2c_\varepsilon)^{-1}$.

Section 3

Examples and convergence

Stochastic Cahn-Hilliard/Allen-Cahn homotopy in 2D

Fix $f(u) = u - u^3$.

CH/AC-homotopy

The limit $\varepsilon \searrow 0$, where $\varepsilon \in (0,1)$, for

$$\partial_t u_\varepsilon = (1 - \varepsilon - \varepsilon\Delta)(\Delta u_\varepsilon + f(u_\varepsilon)) + \sigma_\varepsilon \partial_t W, \quad u_\varepsilon(0) \in L^2(\mathbb{T}^2).$$

For $\varepsilon = 1$, this gives the stochastic Cahn-Hilliard (CH) equation.

The limit $\varepsilon \searrow 0$ with $\sigma_\varepsilon \to \sigma > 0$ would give the stochastic AC equation with space-time white noise which is no longer well-posed in dimension $d \geqslant 2$.

However, for $\sigma_\varepsilon \to 0$, we obtain

$$\partial_t u = \Delta u + f(u) - 3C_0 u,$$

where the constant C_0 is determined by the limit of Z_ε^2 in H^{-1}.

Remarks

Scale of σ_ε

Note that for σ_ε **too small**, we are just in a large deviations principle (LDP)-type regime, where $C_0 = 0$. On the other hand, if σ_ε is **too large**, we are in the renormalization regime, where C_0 has to be replaced by a constant that diverges for $\varepsilon \to 0$.

Products of distributions

Let us remark that we do **neither** need regularity structures, **nor** paracontrolled distributions for our result. Our limit is a deterministic PDE thus we can assume more regularity of the limit u and mixed terms like uZ_ε^2 are always well-defined in the space where Z_ε^2 is well-defined.

Stochastic Allen-Cahn

AC with higher order regularization

The limit $\varepsilon \searrow 0$ for

$$\partial_t u_\varepsilon = -\varepsilon^2 \Delta^2 u_\varepsilon + \Delta u_\varepsilon + f(u_\varepsilon) + \sigma_\varepsilon \partial_t W.$$

AC with regularized noise

The limit $\varepsilon \searrow 0$ for

$$\partial_t u_\varepsilon = \Delta u_\varepsilon + f(u_\varepsilon) + \sigma_\varepsilon \widehat{Q}_\varepsilon \partial_t W.$$

Fourier decomposition

Let $\{e_k\}_{k\in\mathbb{Z}^2}$ be a Fourier basis for $L^2(\mathbb{T}^2)$ such that $(1-\varepsilon-\varepsilon\Delta)\Delta e_k = -\lambda_k(\varepsilon)e_k$ with

$$0 \leqslant \lambda_k(\varepsilon) = (1-\varepsilon-\varepsilon|k|^2)|k|^2 \to \infty$$

for $|k| \to \infty$ and $Q_\varepsilon e_k = \alpha_k(\varepsilon)e_k$, where we assume that Q_ε is Fourier diagonal.

We have that

$$Z_\varepsilon(t) = \sum_{k\in\mathbb{Z}^2} \alpha_k(\varepsilon) I_k^{(\varepsilon)}(t) e_k$$

with

$$I_k^{(\varepsilon)}(t) = \int_0^t e^{-(t-s)\lambda_k(\varepsilon)}\, d\beta_k(s),$$

for complex-valued Brownian motions β_k such that $\bar{\beta}_k = \beta_{-k}$, independent up to this identification.

Convergence of Z_ε (CH/AC homotopy in 2D)

For the stochastic CH/AC homotopy in 2D, $\alpha_k(\varepsilon) = \sigma_\varepsilon$.

We may show that for $\varepsilon \searrow 0$,

$$\mathbb{P}\left(\varepsilon^{1/4} \sup_{[0,T]} \|Z_\varepsilon\|_{C^0} > \frac{1}{2}\right) \to 0,$$

$$\mathbb{P}\left(\varepsilon^{1/2} \|Z_\varepsilon\|^6_{L^6([0,T],L^6)} > \gamma\right) \to 0,$$

and

$$\mathbb{P}\left(\|Z_\varepsilon\|^2_{L^p([0,T],H^{-1})} > \gamma\right) \to 0.$$

Convergence of Z_ε^2 (CH/AC homotopy in 2D)

However,

$$Z_\varepsilon^2(t) = \sum_{k\in\mathbb{Z}^2}\sum_{\ell\in\mathbb{Z}^2} I_{k-\ell}^{(\varepsilon)}(t) I_\ell^{(\varepsilon)}(t) e_k.$$

To get the convergence

$$Z_\varepsilon^2 - C_\varepsilon \to 0 \quad \text{in } L^2(\Omega; L^2([0,T], H^{-1}))$$

we need (note that $\alpha_k(\varepsilon) = \sigma_\varepsilon$)

$$C_\varepsilon = \sum_{k\neq 0} \frac{\sigma_\varepsilon^2}{2\lambda_k(\varepsilon)}$$

in order to handle the divergent constant terms.

Convergence of Z_ε^2(CH/AC homotopy in 2D)

This divergent term matches the zeroth Fourier mode of Z_ε^2. Due to Gaussian cancellations, all other Fourier modes of Z_ε^2 remain finite in the limit $\varepsilon \searrow 0$. See for example [Da Prato, Debussche (2003)], [Blömker, Romito (2013)].

We thus define our renormalizing constant to be

$$C_0 = \lim_{\varepsilon \searrow 0} \sigma_\varepsilon^2 \sum_{k \neq 0} \frac{1}{2\lambda_k(\varepsilon)},$$

and prove that

$$\mathbb{E}\|Z_\varepsilon^2 - C_0\|^2_{L^2([0,T],H^{-1})} \to 0 \quad \text{as} \quad \varepsilon \searrow 0.$$

Thus we need $\sigma(\varepsilon) \sim 1/\log(\varepsilon^{-1})$ in order to have a nontrivial limit.

If $\sigma(\varepsilon) \ll 1/\log(\varepsilon^{-1})$, then $C_0 = 0$ and we have an LDP.

If $\sigma(\varepsilon) \gg 1/\log(\varepsilon^{-1})$, we get triviality as in [Hairer, Ryser, Weber (2012)] that is, $u \equiv 0$ and $u_\varepsilon \approx Z_\varepsilon \to 0$ in H^{-1}, but diverges in L^2.

CH/AC homotopy in 2D

Theorem

For the CH/AC-homotopy with periodic boundary conditions on a two dimensional domain perturbed by space-time white noise of strength

$$\sigma_\varepsilon \sim \frac{1}{\log(\varepsilon^{-1})},$$

supposing that $\|u_\varepsilon(0) - u(0)\|_{L^2} \to 0$ *in probability, we obtain that*

$$\|u_\varepsilon - u - Z_\varepsilon\|_{L^\infty([0,T],L^2)} \to 0 \qquad \textit{in probability,}$$

where $u \in L^\infty([0,T], H^1) \cap L^2([0,T], H^2)$ *is a solution to*

$$\partial_t u = \Delta u + u - u^3 - 3C_0 u.$$

Possible future topics

- Other boundary conditions on a bounded domain,
- Other nonlinearities than cubic, e.g. quadratic or logarithmic,
- Precise asymptotics for σ_ε and convergence of Z_ε^2 in negative fractional Sobolev spaces replacing H^{-1},
- Nonlinear reaction diffusion systems and cross-diffusion systems, as e.g. the *additive noise Gray-Scott system*, for $c_1, c_2, \chi, \sigma_\varepsilon^1, \sigma_\varepsilon^2, \varepsilon > 0$,

$$du = [c_1 \Delta u - \chi u v^2]\, dt + \sigma_\varepsilon^1\, dW^1,$$

$$dv = [c_2 \Delta v + u v^2]\, dt + \sigma_\varepsilon^2\, dW^2,$$

  where stability of the nonlinearity in zero seems crucial.
- Convergence of invariant measures,
- Multiplicative noise case as for instance the generalized parabolic Anderson model — may need approach of paracontrolled distributions.

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Thank you for your attention!