Variability of paths and differential equations with BV-coefficients*

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Contents

Section 1

Motivation

Stieltjes (S)DEs

Consider the pathwise (stochastic) differential equation:

$$dX_t = \sigma(X_t)\, dY_t, \quad X_0 = x_0,$$

in other words

$$X_t = x_0 + \int_0^t \sigma(X_s)\, dY_s. \tag{1}$$

If $\sigma : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous and $Y \in C([0,T]) \cap BV([0,T])$, (1) admits a unique solution $X \in C([0,T])$.

Here, the integral in (1) is interpreted as a *Lebesgue-Stieltjes integral*, as there exists a signed Radon measure μ such that $Y_t = \int_0^t \mu(ds)$ and thus $dY = \mu(dt)$.

Integration against paths

Fix a finite time horizon $T > 0$. Let $Y : [0, T] \to \mathbb{R}$ be a continuous path (signal).

For coefficients $F : \mathbb{R}^2 \to \mathbb{R}$, assumed linear in the second variable, consider the ODE

$$\dot{X}_t = F(X_t, \dot{Y}_t), \quad X_0 = x_0,$$

where $\dot{Y}$ is interpreted as *noise* or *control*.

Usually (in stochastic analysis), Y is not differentiable, so we consider instead

$$X_t = x_0 + \int_0^t \sigma(X_s)\, dY_s. \tag{1}$$

If $\sigma : \mathbb{R} \to \mathbb{R}$ is **Lipschitz continuous** and $Y \in C([0, T]) \cap BV([0, T])$, (1) admits a unique solution.

Here, the integral in (1) is intepreted as *Lebesgue-Stieltjes integral*, as there exists a signed Radon measure μ such that $Y_t = \int_0^t \mu(ds)$ and thus $dY = \mu(dt)$.

Disclaimer (stochastic case)

A typical path (the collection of typical paths has probability one) B of a Brownian motion (independent Gaussian increments) satisfies just $B \in C^{\frac{1}{2}-\varepsilon}([0,T])$ for any $\varepsilon > 0$ and at the same time $B \notin BV([0,T])$.
One can prove that there exists no linear theory of integration extending Stieltjes intergration that is rich enough to handle Brownian paths in a deterministic fashion.

The most popular workaround is the theory of Itô-integration which relies on martingale theory and an L^2-theory of 2-variation (not pathwise).

The Itô integral admits an extension of the change of variable formula (the Itô formula), stated in a simple case:

$$B_t^2 = 2\int_0^t B_s\,dB_s + t.$$

Other approaches:

- Skorokhod integration ($\longrightarrow$ related to the theory of Gaussian and white noise distributions)
- Rough path theory ($\longrightarrow$ estimation of iterated integrals of paths)
- Regularity structures / paracontrolled distributions (Hairer, Gubinelli, Perkowski, ...)

Pathwise integrals $\int_0^t X_s\, dY_s$

			
Thomas J. Stieltjes	Henri L. Lebesgue	Laurence C. Young	Martina Zähle
(1856 – 1894)	(1875 – 1941)	(1905 – 2000)	(1950 –)

- Lebesgue-Stieltjes integral: $Y \in C([0,T]) \cap BV([0,T])$, $X \in C([0,T])$;
- Young integral: $\mathrm{Var}_p(Y) < \infty$, $\mathrm{Var}_q(X) < \infty$, $p, q \geqslant 1$ with $\frac{1}{p} + \frac{1}{q} > 1$, where Var_p denotes the *p-variation.*

  Note: $C^\alpha \subset \{\mathrm{Var}_{1/\alpha}(X) < \infty\}$, $\alpha \in (0,1]$;
- Extension by Zähle (1998/2001): $Y \in W_\infty^{1-\alpha}(T-)$, $X \in W_1^\alpha(0+)$, $\alpha \in (0,1)$.

  Note: $C^\beta \subset W_1^\alpha(0+)$, $\beta > \alpha$; $C^\gamma \subset W_\infty^{1-\alpha}(T-)$, $\gamma + \alpha > 1$.

Pathwise Stieltjes (S)DEs

We are considering the equation for paths $X : [0, T] \to \mathbb{R}^n$ and sufficiently regular *drivers* $Y : [0, T] \to \mathbb{R}^n$:

$$X_t = x_0 + \int_0^t \sigma(X_s)\, dY_s, \quad t \in [0, T]. \tag{1}$$

We are interested in the case of non-Lipschitz or even discontinuous coefficients $\sigma \in BV_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^{n\times n})$.

1. We shall give conditions, such that the *composition* $\sigma \circ X$ is a well-defined function in a Hölder space or fractional Sobolev space;
2. we shall give conditions for the existence of (1) as a Lebesgue-Stieltjes/Zähle integral;
3. we aim to study differential systems with BV-coefficients driven by Hölder paths.

SDEs with non-Lipschitz coefficients σ

Let (B_t) be a Brownian motion (Bm). Then the Itô stochastic differential equation

$$X_t = x_0 + \int_0^t \sigma(X_s)\, dB_s, \tag{2}$$

- is pathwise unique if $\sigma \in BV_{\text{loc}}$ and $\sigma \geqslant c > 0$ on compacts, [Nakao (1972)];
- is well-posed for $\sigma \geqslant c > 0$, $|\sigma(x) - \sigma(y)|^2 \leqslant |g(x) - g(y)|$, g increasing and bounded, [Le Gall (1983)];
- has a weak solution if $\sigma^{-1} \in L^2_{\text{loc}}$, [Engelbert, Schmidt (1985)];
- has strong and positive solutions and is pathwise unique if σ is of power type $\sigma(x) = |x|^\alpha$, $\alpha \in (0,1)$, [Bass, Chen (2005)];
- … *more* …

Nualart and Rășcanu (2002) studied (2) with (B_t) replaced by the fractional Brownian motion (fBm) (B_t^H), with Hurst index $H > \frac{1}{2}$ and σ Lipschitz and of linear growth.

Definition

We call a continuous stochastic process $(B_t^H)_{t \geqslant 0}$ an $\mathbb{R}^n$-valued fractional Brownian motion (fBm) with Hurst index $H \in (0,1)$ and starting in $x \in \mathbb{R}^n$ if

1. $B_0^H = x$.
2. $(B_t^H - x)_{t \geqslant 0}$ is centered Gaussian with covariance $\mathbb{E}[(B_t^H - x)^i (B_s^H - x)^j] = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right)\delta_{ij}$.

For $H = \frac{1}{2}$, the fBm is a Bm, as $\mathbb{E}[(B_t^H - x)^i (B_s^H - x)^j] = \min(t,s)\delta_{ij}$. A path of a fractional Brownian motion is in C^α for any $\alpha < H$ with probability one.

[León, Nualart, Tindel, (2017)] proved existence of solutions for fBm and the non-Lipschitz power type $\sigma(x) = |x|^\gamma$, $\gamma \in (0,1)$.

Example in 1D

Consider the one-dimensional case and let $\sigma(x) = \mathbf{1}_{x>a}$, $a \in \mathbb{R}$.

Problem

Usually $\mathbf{1}_{X_t>a}$ is not continuous or of bounded p-variation for any $p \geqslant 1$.

$\Rightarrow$ integral $\int_0^T \mathbf{1}_{X_t>a}\, dX_t$ cannot be defined as *Young integral*, as *Föllmer integral*, or via rough path theory.

Solution

We may use *generalised Lebesgue–Stieltjes integrals* (studied for instance in [Zähle (1998)]). Namely, if $X \in W_0^{\beta,1}$, i.e., belongs to certain time-weighted fractional Sobolev space, and $Y \in W_T^{1-\beta,1}$ for some $\beta \in (0,1)$, then $\int_0^T X_s\, dY_s$ exists.

The notion of *sufficient variability*

Let $X \in C^{\alpha}, Y \in C^{\gamma}$ with $\alpha + \gamma > 1$ and let $\sigma \in BV_{\mathrm{loc}}$. Suppose further that for $\theta \in (0,1)$

$$\sup_{a \in \mathbb{R}} \mathbb{E} \int_0^T |X_t - a|^{-\theta} dt < \infty. \tag{2}$$

Condition (2) guarantees that X is active enough and does not spend too much time on any point a, and in particular, on the bad points / jumps of σ.

Roughly speaking, X will enter the bad region of σ (otherwise situation is boring) but does not stay there very long.

Results can be found in [Chen, Leskelä, Viitasaari, Stoch. Process. Appl. (2019)].

Section 2

Variability of paths and integrals with BV-coefficients

Variability of paths (w.r.t. BV-functions)

Let $\varphi \in BV_{\mathrm{loc}}(\mathbb{R}^n)$, that is, $\varphi \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and the distributional derivatives $D_i\varphi$ are signed Radon measures. Denote by $\|D\varphi\|$ the total variation measure of φ.

Definition

Let $p \in [1,\infty]$, $s \in (0,1)$. We say that $X \in C([0,T])$ is (s,p)*-variable with respect to* φ if there exists a relatively compact open neighborhood $\mathcal{U}$ of $X([0,T])$ such that

$$\int_{\mathcal{U}} |X_\cdot - z|^{-n+1-s} \|D\varphi\|(dz) \in L^p(0,T).$$

Let $V(\varphi,s,p)$ denote the class of continuous paths that are (s,p)-variable w.r.t. φ.

This is a quantitative condition relative to a fixed φ. Clearly, $V(\varphi,s,p) \subset V(\varphi,s,q)$ for $q < p$ and $V(\varphi,s,p) \subset V(\varphi,r,p)$ for $r < s$.

Occupation measure, mutual Riesz energy and Riesz potential

Set

$$\mu := \mu_X^{[0,T]}(B) := \mathcal{L}^1([0,T] \cap X^{-1}(B))$$

as the *occupation measure* of X, in other words,

$$\int_{\mathbb{R}^n} g(x)\,\mu(dx) = \int_0^T g(X_t)\,dt$$

for all bounded Borel functions $g : \mathbb{R}^n \to \mathbb{R}$.

For $p = 1$, variability reads as

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |x-z|^{-n+1-s} \|D\varphi\|(dz)\,\mu(dx) < \infty.$$

Moreover, for $p \in [1,\infty)$, $X \in V(\varphi, s, p)$ if and only if

$$\int_{\mathbb{R}^n} \left(U^{1-s}\|D\varphi\|\right)^p \, d\mu < \infty.$$

Here, $(U^\gamma \nu)(x) := c(\gamma, n) \int_{\mathbb{R}^n} |x-y|^{\gamma-n}\,\nu(dy)$ denotes the *Riesz-potential operator.*

Intuition for variability

The concept of variability in 1D was used and developed before in e.g.:

- [Chen, *Doctoral Dissertation*, Aalto University (2016)]
- [Chen, Leskelä, Viitasaari, Stoch. Process. Appl. (2019)]
- [Torres, Viitasaari, Theor. Probability and Math. Statist. (2023)]

The idea is to capture the 'activity' of paths relative to compositions in a quantitative way and extend this to higher dimensions.

(s, p)-variability excludes 'too bad' behavior at jumps of φ, meaning in particular that Y should be sufficiently disperse at these (lower dimensional) regions.

Upper regularity

Recall that, given $d \geqslant 0$, a Borel measure μ on $\mathbb{R}^n$ is said to be *upper d-regular on a Borel set* $B \subset \mathbb{R}^n$ if there are constants $c > 0$ and $r_0 > 0$ such that

$$\mu(B(x,r)) \leqslant cr^d, \quad x \in B \cap \operatorname{supp} \mu, \quad 0 < r < r_0.$$

We call a function $\varphi \in BV_{\mathrm{loc}}(\mathbb{R}^n)$ *upper d-regular on B* if $\|D\varphi\|$ is *upper d-regular on* B.

Note that upper d-regularity with $d > n - 1 + s$ implies that φ has a unique Borel version being Hölder continuous of order s.

Upper regularity of paths

We also consider an upper regularity condition for paths. If for given $s > 0$ and $B \subset \mathbb{R}^n$ Borel a path $Y : [0, T] \to \mathbb{R}^n$ satisfies

$$\sup_{x \in Y([0,T]) \cap B} \int_0^T |Y_t - x|^{-s} dt < +\infty \tag{5}$$

then there are constants $c > 0$ and $r_0 > 0$ such that

$$|\{t \in [0, T] : |Y_t - Y_u| < r\}| \leqslant c r^s, \quad u \in \{\tau \in [0, T] : Y_\tau \in B\}, \quad 0 < r < r_0. \tag{6}$$

If (6) holds with some $d > s$ in place of s, then (5) holds.

Examples

Extreme behavior of φ

If $\|D\varphi\|$ is *upper regular* of order $d > n - 1 + s$, then any path X is $(s, 1)$-variable. This on the other hand forces φ to be Hölder continuous (and hence more "classical" approaches apply).
In particular, if φ is locally Lipschitz, then for any $s \in (0, 1)$ any continuous path is $(s, 1)$-variable.

Extreme behavior of the path X

If the occupation measure μ of X is *upper regular* of order $d > n - 1 + s$, then X is $(s, 1)$-variable w.r.t. any φ. This should be compared to condition (2) in the one-dimensional case, where

$$\sup_{a \in \mathbb{R}} \mathbb{E} \int_0^T |X_t - a|^{-s} dt < \infty \tag{2}$$

implies that any bounded variation function φ is good enough.

Other examples can be constructed:

- Splitting the space into regions where either μ or $\|D\varphi\|$ is upper regular;
- "in between" cases where neither of the measures is (sufficiently) upper regular.

From [Hinz, T, Viitasaari, Electronic J. Probab. (2022)]:

Fractional Brownian motion

If $X = B^H$ is the n-dimensional fBm with Hurst index $0 < H < 1$, its occupation measure μ is upper d-regular $\mathbb{P}$-a.s. if $0 < d \leqslant n$ is such that $H < \frac{1}{d}$. This implies that B^H is $\mathbb{P}$-a.s. $(s,1)$-variable w.r.t. any $\varphi \in BV(\mathbb{R}^n)$ if $n - 1 + s < \frac{1}{H}$. If $\frac{1}{H} < n - 1 + s$, we need the negative moment condition

$$\int_{\mathbb{R}^n} |x|^{-n+1-s+\frac{1}{H}} \|D\varphi\|(dx) < +\infty.$$

The $(s,2)$-variability can be rephrased in terms of weighted Wolff potentials of $\|D\varphi\|$.

Lévy processes

Let $n \geqslant 2$ and let X be an isotropic α-stable Lévy process, where $0 < \alpha < 2$. X is $\mathbb{P}$-a.s. $(s,1)$-variable for $\varphi \in BV(\mathbb{R}^n)$ if

$$\int_{\mathbb{R}^n} |x|^{-n+1-s+\alpha} \|D\varphi\|(dx) < +\infty.$$

Fractional Sobolev spaces

Recall the definition of the *fractional Sobolev space* $W^{\beta,p}(0,T)$:

By $W^{\beta,p}(0,T)$, we denote the space of measurable functions $f:(0,T)\to\mathbb{R}$ such that

$$\|f\|_{W^{\beta,p}(0,T)} := \left(\|f\|^p_{L^p(0,T)} + \int_0^T\int_0^T \frac{|f(t)-f(s)|^p}{|t-s|^{1+\beta p}}\,ds\,dt\right)^{1/p} < \infty,$$

if $p\in[1,\infty)$. We also set for $p=\infty$

$$\|f\|_{W^{\beta,\infty}(0,T)} := \|f\|_{L^\infty(0,T)} + \operatorname*{ess\,sup}_{t\in[0,T]}\int_0^t \frac{|f(t)-f(s)|}{|t-s|^{1+\beta}}\,ds < \infty.$$

Define also the weighted space $W_0^{\beta,p}(0,T)$ by

$$\|f\|_{W_0^{\beta,p}(0,T)} := \left(\int_0^T \frac{|f(t)|^p}{t^{\beta p}}\,dt + \int_0^T\int_0^T \frac{|f(t)-f(s)|^p}{|t-s|^{1+\beta p}}\,ds\,dt\right)^{1/p} < \infty.$$

Theorem ([Hinz, T, Viitasaari, AIHP (B) Probab. Stat. (2023)])

Suppose that $\varphi \in BV_{\mathrm{loc}}(\mathbb{R}^n)$ and that $X \in C^{\alpha}([0,T],\mathbb{R}^n)$ is $(s,1)$-variable with respect to φ for some $s \in (0,1)$.

1. *For any $0 < \beta < \alpha s$ the composition $\varphi \circ X$ is an element of $W_0^{\beta,1}(0,T)$.*
2. *If, in addition, $Y \in C^{\gamma}([0,T])$ and $1 - \alpha s < \gamma$ then for any $t \in [0,T]$ the integral $\int_0^{\cdot} \varphi(X_u)dY_u$ exists in the sense of [Zähle (1998)], [Schneider, Zähle (2019)].*
3. *If, moreover, X is (s,p)-variable with respect to φ for some $p \in (1,+\infty]$ then for any $0 < \beta < \alpha s$ we have $\varphi \circ X \in W^{\beta,p}(0,T)$.*
4. *If, in addition, $Y \in C^{\gamma}([0,T])$ with $\frac{1}{p} < 1 - \beta < \gamma$, then*

$$\int_0^{\cdot} \varphi(X_u)dY_u \in C^{1-\beta-1/p}([0,T])$$

and there exists a constant $c > 0$ such that

$$\left\| \int_0^{\cdot} \varphi(X_u)dY_u \right\|_{C^{1-\beta-1/p}([0,T])} \leqslant c\, \|\varphi \circ X\|_{W^{\beta,p}(0,T)} \|Y\|_{C^{\gamma}([0,T])}.$$

Theorem ([Hinz, T, Viitasaari, Electronic J. Probab. (2022)])

Let $\Omega \subset \mathbb{R}^k$ be a bounded domain. Let $\varphi \in BV(\mathbb{R}^n)$, $u : \Omega \to \mathbb{R}^n$ a measurable function, $s, \theta \in (0,1)$, $1 \leqslant p < +\infty$ and $1 \leqslant q \leqslant +\infty$. Suppose that

$$u \in W^{\theta,q}(\Omega, \mathbb{R}^n) \cap V(\varphi, s, p).$$

1. *If $r \geqslant 1$ is such that $\frac{1}{p} + \frac{s}{q} \leqslant \frac{1}{r}$ and $\beta < s\theta$, then $\varphi \circ u \in W^{\beta,r}(\Omega)$ and in particular,*

$$[\varphi \circ u]_{\beta,r} \leqslant c\,[u]_{\theta,q}^{s}\, \| U^{1-s}\, \|D\varphi\|\,(u(\cdot))\|_{L^p(\Omega)}$$

   with a constant $c > 0$ depending only on n, s, p, q, r, θ, β and Ω.

2. *If $\frac{1}{p} + \frac{s}{q} < s\theta$, then for any $\theta' \leqslant \theta - \frac{1}{q}$ we have $u \in W^{\theta',\infty}(\Omega, \mathbb{R}^n)$, and for any $\beta < s\theta - \frac{1}{p} - \frac{s}{q}$ we have $\varphi \circ u \in W^{\beta,\infty}(\Omega)$.*

Here, we denote the Gagliardo seminorm by

$$[f]_{\theta,p} := \left(\int_\Omega \int_\Omega \frac{|f(y) - f(z)|^p}{|y-z|^{k+\theta p}}\, dy\, dz \right)^{\frac{1}{p}}.$$

The change of variable formula

Theorem ([Hinz, T, Viitasaari, AIHP (B) Probab. Stat. (2023)])

Let $F \in W^{1,1}_{\text{loc}}(\mathbb{R}^n)$ be such that $\partial_k F \in BV_{\text{loc}}(\mathbb{R}^n)$ for $k = 1, \dots, n$. If $X \in C^\alpha([0,T], \mathbb{R}^n)$ with $\alpha > \frac{1}{2}$ is a path which is $(s,1)$-variable w.r.t. each $\partial_k F$ for some for some $s \in \left(\frac{1-\alpha}{\alpha}, 1\right)$, then we have

$$F(X_t) = F(x_0) + \sum_{k=1}^{n} \int_0^t \partial_k F(X_s) dX_s^k \tag{3}$$

for dt-a.e. $t \in [0,T]$, provided that $x_0 \in \mathbb{R}^n \setminus S_F$.

If, in addition, F is continuous, then (3) holds for all $t \in [0,T]$ and no matter where $x_0 \in \mathbb{R}^n$ is located.

Approximation by Riemann-Stieltjes sums

Theorem ([Hinz, T, Viitasaari, AIHP (B) Probab. Stat. (2023)])

Let $\varphi \in BV_{\mathrm{loc}}(\mathbb{R}^n)$, let $X \in C^\alpha([0,T],\mathbb{R}^n)$ be a path which is (s,p)-variable with respect to φ for some $s \in (0,1)$ and $p \in (\frac{1}{\alpha s}, +\infty]$. Then $\varphi \circ X$ is Hölder continuous of any order less than $\alpha s - \frac{1}{p}$. If in addition $Y \in C^\gamma([0,T])$ for some $\gamma > 1 - \alpha s + \frac{1}{p}$, where $\frac{1}{\infty} := 0$, then both the generalized Lebesgue-Stieltjes integral $\int_0^T \varphi(X_u)\, dY_u$ as in (2) and the Riemann-Stieltjes integral of $\varphi(X)$ w.r.t. Y over $[0,T]$ exist and agree.

If, in this case, we are given $0 < \varepsilon < \alpha s - \frac{1}{p} - 1 + \gamma$, a refining sequence $(\tau_k)_{k \geqslant 1}$ of finite partitions $\tau_k = \{0 = t_0^{(k)} < t_1^{(k)} < \ldots < t_{r_k}^{(k)} = T\} \subset [0,T]$ and $\xi_i^{(k)} \in [t_{i-1}^{(k)}, t_i^{(k)}]$, then we have

$$\left| \int_0^T \varphi(X_u)\, dY_u - \sum_{i=1}^{r_k} \varphi(X_{\xi_i^{(k)}}) \left(Y_{t_i^{(k)}} - Y_{t_{i-1}^{(k)}} \right) \right| \leqslant c|\tau_k|^{\alpha s - \frac{1}{p} - 1 + \gamma - \epsilon}$$

for all k, where $c > 0$ is a constant depending on α, γ, s, p and Y.

Existence of rough integrals

Set $\gamma^{\theta,r}(t) := tX_r + (1-t)X_\theta$, and let

$\Delta := \{(s,t) \subset [0,T] \times [0,T] : \quad 0 \leqslant s \leqslant t \leqslant T\}$.

Theorem ([Hinz, T, Viitasaari, arXiv:2407.06907])

Suppose that $\frac{1}{3} < \beta < \frac{1}{2}$, $(X, Y, X \otimes Y) \in C^\beta([0,T]) \times C^\beta([0,T]) \times C^{2\beta}(\Delta)$, $\varphi \in \mathrm{Lip}(\mathbb{R}^m; \mathbb{R}^d)$ *with* $\partial_i \varphi_j \in BV_{\mathrm{loc}}(\mathbb{R}^m)$ *for all* i, j *and*

$$\frac{1}{\beta} - 2 < s < 1.$$

If X *is* $(s, 1)$*-variable w.r.t. all* $\partial_i \varphi_j$ *and in addition*

$$\int_0^T \int_0^T |r - \theta|^{\beta(1+s)-\alpha-1} \int_0^1 U^{1-s} \|D\partial_i \varphi_j\|(\gamma^{\theta,r}(t)) t^s \, dt \, d\theta \, dr < +\infty$$

Theorem ([Hinz, T, Viitasaari, arXiv:2407.06907] (cont'd))

Then $\int_0^T \varphi(X)\,dY$ *exists as a Hu-Nualart-Zähle*[†] *rough integral with*

$$
\begin{aligned}
&\Big|\int_0^T \varphi(X)\,dY\Big| \\
&\lesssim \sum_{j=1}^{d} \Big\{ \|\varphi_j(X)\|_{L^1(0,T)} + \mathrm{Lip}(\varphi_j)[X]_\beta \\
&\quad + \sum_{i=1}^{m} \int_0^T \int_0^T |r-\theta|^{\beta(1+s)-\alpha-1} \int_0^1 U^{1-s}\|D\partial_i\varphi_j\|(\gamma^{\theta,r}(t))t^s\,dt\,d\theta\,dr\,[X]_\beta^{1+s} \\
&\quad + \sum_{i=1}^{m} \|U^{1-s}\,\|D\partial_i\varphi_j\|\,(X)\|_{L^1(0,T)}\,[X]_\beta^{1+s}\Big\}[Y^j]_\beta \\
&\quad + \sum_{j=1}^{d}\sum_{i=1}^{m}\Big\{\|\partial_i\varphi_j(X)\|_{L^1(0,T)} + \|U^{1-s}\|D\partial_i\varphi_j\|(X)\|_{L^1(0,T)}[X]_\beta^s\Big\}[(X\otimes Y)^{i,j}_{\cdot,T}]_{2\beta}.
\end{aligned}
$$

† [Hu, Nualart, Trans. Amer. Math. Soc. (2009)].

Existence of rough integrals (cont'd)

For the fractional Brownian motion (fBm), the variability condition along line segments can be verified in dimensions 1 and 2.

Proposition ([Hinz, T, Viitasaari, arXiv:2407.06907])

For $\gamma^{\theta,r}(t) := tX_r + (1-t)X_\theta$, and X fBm on $[0,T]$ with Hurst index $H \in (\frac{1}{3}, \frac{1}{2}]$, we have that

$$\int_0^T \int_0^T |r-\theta|^{\beta(1+s)-\alpha-1} \int_0^1 U^{1-s} \|D\partial_i \varphi_j\|(\gamma^{\theta,r}(t)) t^s \, dt \, d\theta \, dr < +\infty$$

holds $\mathbb{P}$-a.s. in dimensions $m = 1, 2$.

Section 3

Differential systems with BV-coefficients

Doss' transform [Lamperti (1964), Doss (1977), Sussmann (1978), Yamato (1979)]

Let (B_t) be a one-dimensional Bm. Consider the Stratonovich SDE

$$\circ dX_t = b(X_t)\,dt + \sigma(X_t) \circ dB_t, \quad X_0 = x, \tag{SDE}$$

with $b \in \mathrm{Lip}(\mathbb{R}^n, \mathbb{R}^n)$, $\sigma \in C_b^2(\mathbb{R}^n, \mathbb{R}^n)$. Consider the ODE system

$$\frac{d}{dt}F = \sigma(F), \quad F(0) = x. \tag{ODE}$$

If $b \equiv 0$, then $X_t := F(B_t, x)$ is a solution to (SDE), where $(t, x) \mapsto F(t, x)$ is the unique solution to (ODE), that is, the *flow* of the vector field σ.

If $b \not\equiv 0$, then $X_t = F(B_t, C_t)$ is a (unique) strong solution to (SDE), where (C_t) solves the pathwise random ODE

$$\frac{d}{dt}C_t = \frac{\partial F}{\partial x}(B_t, C_t)^{-1} b(F(B_t, C_t)).$$

Differential systems

Consider

$$X_t = x_0 + \int_0^t \sigma(X_u)\, dY_u, \quad t \in [0, T], \tag{3}$$

where $T > 0$, $x_0 \in \mathbb{R}^n$, $\sigma : \mathbb{R}^n \mapsto \mathbb{R}^{n\times n}$ is a coefficient function $\sigma = (\sigma_{jk})_{1\leqslant j,k\leqslant n}$ and $Y = (Y^1, \ldots, Y^n) : [0, T] \to \mathbb{R}^n$ is a given Hölder path.

As usual, (3) is to be understood in the sense that all components X^j of $X = (X^1, \ldots, X^n) : [0, T] \to \mathbb{R}^n$ should satisfy the equations

$$X_t^j = x_0^j + \sum_{k=1}^n \int_0^t \sigma_{jk}(X_t^1, \ldots, X_t^n)\, dY_t^k,$$

where $x_0 = (x_0^1, \ldots, x_0^n)$.

We are interested in the case that the components of σ are BV_{loc}.

Rough idea: Solve $\nabla f = \sigma(f)$, and define $X := f(Y)$ for the driver Y and use the change of variable formula! σ has to be invertible and "curl-free".

Variability solutions

We consider the following notion of a solution to (3).

Definition

Let $\sigma = (\sigma_{jk})_{1 \leqslant j,k \leqslant n}$ be such that $\sigma_{jk} \in BV_{\text{loc}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ for all j and k and let $Y \in C^\gamma([0,T],\mathbb{R}^n)$. A path $X : [0,T] \to \mathbb{R}^n$ is called a *variability solution for σ and Y started at $x_0 \in \mathbb{R}^n$* if

1. $X_0 = x_0$,
2. the path X is in $C^\alpha([0,T],\mathbb{R}^n)$ and also in $V(\sigma, s, 1)$ for some $s \in (\frac{1-\gamma}{\alpha}, 1)$,
3. X satisfies (3).

The second part of condition 2. is in principle needed to rule out trivial solutions $X \equiv c$ with such that $D\sigma$ has a singularity in c.

One-dimensional case

Theorem (Garzón, León, Torres (2017); Torres, Viitasaari (2023); Hinz, T, Viitasaari (2023))

Let $\sigma \in BV_{\mathrm{loc}}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ be nonnegative $\mathcal{L}^1$-a.e. and such that $\frac{1}{\sigma} \in L^1_{\mathrm{loc}}(\mathbb{R})$.

1. *The function $g(x) = \int_0^x \frac{dz}{\sigma(z)}$, $x \in \mathbb{R}$, is absolutely continuous and strictly increasing on $\mathbb{R}$. Its inverse $f := g^{-1}$ is Lipschitz and satisfies $\sigma(f) = f'$ $\mathcal{L}^1$-a.e. on $\mathbb{R}$.*
2. *Let $s \in (0,1)$, $\gamma \in (\frac{1}{1+s}, 1)$, $Y \in C^\gamma([0,T])$ with $Y_0 = 0$ and $\mathring{x} \in \mathbb{R}$. Let $-\infty \leqslant a < b \leqslant +\infty$. Suppose that Y satisfies (5) for $B = (g(a), g(b))$ or that σ is upper d-regular on (a,b) with $d > s$, and similarly for $\mathbb{R} \setminus (g(a), g(b))$ and $\mathbb{R} \setminus (a,b)$, respectively.*

Then the function

$$X_t = f(Y_t + g(\mathring{x})), \quad t \in [0,T],$$

is a variability solution with $X \in C^\gamma([0,T]) \cap V(\sigma, s)$ for σ and Y started at $\mathring{x}$.

Multi-dimensional case (via Doss' transform)

Assume the following:

1. $\sigma = (\sigma_{jk})_{1 \leqslant j,k \leqslant n}$ with components $\sigma_{jk} \in BV_{\mathrm{loc}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ for all j and k,
2. $\det(\sigma) > \varepsilon$, $\mathcal{L}^n$-a.e. on $\mathbb{R}^n$ for some $\varepsilon > 0$,
3. $(-\infty, 0) \cap \mathrm{spec}(\sigma(x)) = \varnothing$ for $\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n$,
4. $D_i \sigma^{-1}_{kj} - D_j \sigma^{-1}_{ki} = 0$ (“curl-free”) in the sense of tempered distributions.

Then the stationary PDE

$$\nabla f = \sigma(f). \tag{4}$$

has a bi-Lipschitz solution.

Main existence result (via Doss' transform)

Theorem ([Hinz, T, Viitasaari, AIHP (B) Probab. Stat. (2023)])

Suppose that σ is as above and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a bi-Lipschitz function which solves (4). Let $s \in (0,1)$, $\gamma \in (\frac{1}{1+s}, 1)$, let $Y = (Y_t)_{t\in[0,T]}$ be an $\mathbb{R}^n$-valued stochastic process with $Y_0 = 0$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with paths $\mathbb{P}$-a.s. Hölder continuous of order γ, and let $\mathring{x} \in \mathbb{R}^n$. Suppose that there are $\varepsilon \in (0, 1-s)$, $c > 0$, and $\delta \in (0, n-1+s+\varepsilon)$ such that

$$\mathbb{E}\int_0^T |Y_t - x|^{-n+\varepsilon} dt < c|x|^{-n+\delta}, \quad x \in \mathbb{R}^n,$$

and for all j and k we have

$$\int_{\mathbb{R}^n} |x - \mathring{x}|^{-n+1-s-\varepsilon+\delta} \, \|D\sigma_{jk}\| \, (dx) < +\infty. \tag{5}$$

Then for $\mathbb{P}$-a.e. $\omega \in \Omega$ the path

$$X_t(\omega) = f(Y_t(\omega) + f^{-1}(x)), \quad t \in [0, T],$$

is a nontrivial solution $X \in C^\gamma([0,T], \mathbb{R}^n) \cap V(\sigma, s)$ for σ and Y started at $\mathring{x}$.

The moment condition (5) excludes a too bad behavior of σ at the starting point $\mathring{x}$ at time $t = 0$.

Uniqueness

To get uniqueness, we assume that for all i, j, and k we have the *"curl-free" condition*

$$D_i\sigma_{kj}^{-1} - D_j\sigma_{ki}^{-1} = 0 \tag{6}$$

in the sense of tempered distributions.

Theorem ([Hinz, T, Viitasaari, AIHP (B) Probab. Stat. (2023)])

Suppose that $\sigma = (\sigma_{jk})_{1\leqslant j,k\leqslant n}$ satisfies the assumptions of the existence theorem and (6) and that $Y \in C^\gamma([0,T],\mathbb{R}^n)$ for some $\gamma \in (0,1)$ and $x_0 \in \mathbb{R}^n$. Then there exists at most one variability solution of Hölder order greater $\frac{1}{2}$ for σ and Y started at x_0.

2D example

Doss' method is quite restrictive, as seen in the following 2D example. Consider

$$dX_t = g(t, X_t)\,dt + h(t, X_t)\,dY_t, \quad t \geqslant 0,$$

with Hölder continuous driver Y and $g, h \in BV_{\text{loc}}(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, $h \geqslant \varepsilon > 0$. Writing $dZ_t = dt$, we need to solve the system

$$d\widehat{X} = \sigma(\widehat{X})\,d\widehat{Y},$$

where $\widehat{X} := (Z, X)^T$, and $\widehat{Y} := (t, Y)^T$ and

$$\sigma := \begin{pmatrix} 1 & 0 \\ g & h \end{pmatrix}.$$

However, the "curl-free" condition is satified if and only if

$$\partial_y g(t, y) = \partial_t h(t, y), \quad \forall t, y \in \mathbb{R}, \quad \text{in the sense of Schwartz distributions.}$$

Wrap-up

We have seen that:

- the notion of *variability* quantifies when a given path is active enough on bad points of a BV-function to ensure membership of the composition $\varphi \circ X$ in fractional Sobolev classes.
- We may also define generalized Lebesgue-Stieltjes integrals of the compositions $\int_0^t \varphi(X_u)\, dY_u$.
- Once the integral is defined, we may use Doss' transform to solve differential systems $dX = \sigma(X)\, dY$.

Side dishes

- More general results for Borel paths and for fractional Sobolev spaces;
- regularity properties of the generalized Lebesgue-Stieltjes integrals;
- conditions for the existence of Riemann-Stieltjes approximations;
- Fourier analysis of variability and occupation measures of stochastic processes.

Outlook

Future topics

- detailed analysis of variablity (also on the Fourier side) using e.g. Wolff potentials and weighted potential theory for important classes of stochastic processes (Gaussian, Lévy, ...);
- **Dream result:** (Stochastic) Peano- or Picard theorems for equations with BV-coefficients
  $\longrightarrow$ deeper understanding of variability of (iterated) integral operators;
- Sewing lemma?
- Applications to fractional PDEs of semilinear Nemytskii type or of fractional porous medium type.

References

Zh. Chen, L. Leskelä, L. Viitasaari, *Pathwise Stieltjes integrals of discontinuously evaluated stochastic processes*, Stochastic Processes and their Applications 129 (2019) 2723–2757.

A. Figalli, *Existence and uniqueness of martingale solutions for SDEs with rough or degenerate coefficients*, J. Funct. Anal. **254** (2008), 109–153.

J. Garzón, J.A. Leon, S. Torres, *Fractional stochastic differential equation with discontinuous diffusion*, J. Stoch. Anal. Appl. **35**(6) (2017), 1113–1123.

L. V. Kovalev, J. Onninen, and K. Rajala. *Invertibility of Sobolev mappings under minimal hypotheses*. Ann. Inst. H. Poincaré Anal. Non Linéaire, **27**(2) (2010) 517–528.

E. Schneider, M. Zähle, *Forward integrals and SDE with fractal noise*, In: *Horizons of Fractal Geometry and Complex Dimensions*, Contemporary Mathematics Volume **731**, 2019, 279–302.

T. Tao, J. Wright, *L^p improving bounds for averages along curves*, J. Amer. Math. Soc. **16**(3) (2003), 605–638.

S. Torres, L. Viitasaari, *Stochastic differential equations with discontinuous diffusion coefficients*, Theor. Probability and Math. Statist. **109** (2023), 159–175.

L.C. Young, *An inequality of the Hölder type, connected with Stieltjes integration*, Acta Math. **67** (1936), 251–282.

M. Zähle, *Integration with respect to fractal functions and stochastic calculus I*, Probab. Th. Relat. Fields **111** (1998), 333–374.

M. Zähle, *Integration with respect to fractal functions and stochastic calculus II*, Math. Nachr. **225**(1) (2001), 145–183.

Thank you for your attention!

Section 4

Backup

p-variation

Denote by $\tau \subset [0,T]$ a *partition* $\tau := \{0 \leqslant t_0 < t_1 < \ldots < t_{r-1} < t_r \leqslant T\}$.

Definition

Let $Y : [0,T] \to \mathbb{R}^n$ be a continuous path. For $p \geqslant 1$, consider

$$\mathrm{Var}_p(Y) := \left[\sup_{\tau \subset [0,T]} \sum_{i=0}^{r-1} |Y_{t_i} - Y_{t_{i+1}}|^p \right]^{1/p}$$

the *p-variation*.

A continuous path Y is in $BV([0,T])$ if and only if it satisfies $\mathrm{Var}_1(Y) < \infty$.

Fact

A path $Y \in C^\alpha([0,T])$ *with* $0 < \alpha \leqslant 1$ *satisfies* $\mathrm{Var}_{\frac{1}{\alpha}}(Y) < \infty$.

Young integrals

Denote the mesh of a partition $\tau \subset [0,T]$ by $|\tau| := \max(t_0, t_1 - t_0, \ldots, t_r - t_{r-1}, T - t_r)$.

Theorem ([Young (1936)])

Let $p, q \geqslant 1$ such that $\frac{1}{p} + \frac{1}{q} > 1$. Let $Y : [0,T] \to \mathbb{R}^n$ be a continuous path with $\mathrm{Var}_p(Y) < \infty$ and let $X : [0,T] \to (\mathbb{R}^n)^N$ be a continuous path with $\mathrm{Var}_q(X) < \infty$. Then, for each $t \in [0,T]$, the limit

$$\int_0^t X_s \, dY_s := \lim_{|\tau| \to 0, \tau \subset [0,t]} \sum_{i=0}^{r-1} X_{t_i} (Y_{t_{i+1}} - Y_{t_i})$$

exists. As a function of t this limit is a continuous map from $[0,T]$ to $\mathbb{R}^N$ with finite p-variation and there exists a constant $C = C(p,q) > 0$ such that

$$\mathrm{Var}_p\left(\int_0^\cdot (X_s - X_0)\, dY_s\right) \leqslant C \,\mathrm{Var}_q(X)\,\mathrm{Var}_p(Y).$$

Peano's and Picard's theorems

Let us return to well-posedness of

$$dX = f(X)\,dY, \quad X_0 = x_0. \tag{1}$$

Theorem (Peano)

Let $1 \leqslant p < 2$ and let $p-1 < \gamma \leqslant 1$. Assume that $Y : [0,T] \to \mathbb{R}^n$ is continuous with $\mathrm{Var}_p(Y) < \infty$. Let $f : \mathbb{R}^N \to (\mathbb{R}^n)^N$ be in $C^\gamma(\mathbb{R}^N, (\mathbb{R}^n)^N)$. Then, for any $x_0 \in \mathbb{R}^N$, (1) admits a continuous solution $X : [0,T] \to \mathbb{R}^N$ of finite p-variation.

Theorem (Picard)

Let $1 \leqslant p < 2$ and let $p < \gamma$. Assume that $Y : [0,T] \to \mathbb{R}^n$ is continuous with $\mathrm{Var}_p(Y) < \infty$. Let $f : \mathbb{R}^N \to (\mathbb{R}^n)^N$ be in $C^\gamma(\mathbb{R}^N, (\mathbb{R}^n)^N)$. Then, for any $x_0 \in \mathbb{R}^N$, (1) admits a unique continuous solution $X : [0,T] \to \mathbb{R}^N$ of finite p-variation.

Definition

A function $\varphi \in L^1_{\text{loc}}(\mathbb{R}^n)$ is said to have an *approximate limit* at $x \in \mathbb{R}^n$ if there exists $\lambda_\varphi(x) \in \mathbb{R}$ such that

$$\lim_{r \to 0} \frac{1}{B(x,r)} \int_{B(x,r)} |\varphi(y) - \lambda_\varphi(x)| \, dy = 0.$$

In this situation, the unique value $\lambda_\varphi(x)$ is called the *approximate limit* of φ at x.

The set of points $x \in \mathbb{R}^n$ for which this property does not hold is called *approximate discontinuity set* (or *exceptional set*) and is denoted by S_φ.

Definition

If $\widetilde{\varphi}$ is a representative of $\varphi \in L^1_{\text{loc}}(\mathbb{R}^n)$ then a point $x \notin S_\varphi$ with $\widetilde{\varphi}(x) = \lambda_\varphi(x)$ is called a *Lebesgue point* of $\widetilde{\varphi}$. A Borel function $\widetilde{\varphi} : \mathbb{R}^n \to \mathbb{R}$ is called *Lebesgue representative* if

$$\widetilde{\varphi}(x) = \lambda_\varphi(x), \quad x \in \mathbb{R}^n \setminus S_\varphi.$$

If $\varphi \in W^{1,1}(\mathbb{R}^n)$, then $\mathcal{H}^{n-1}(S_\varphi) = 0$. On the other hand, if $\varphi \in BV(\mathbb{R}^n)$, φ can have jumps with support in $J_\varphi \subset S_\varphi$ such that $\mathcal{H}^{n-1}(J_\varphi) > 0$.

Sets of finite perimeter

Suppose that $O \subset \mathbb{R}^n$ is a smooth domain with $\mathcal{H}^{n-1}(\partial O) < +\infty$. The function 1_O is in $BV(\mathbb{R}^n)$ and O has finite perimeter

$$P(O, \mathbb{R}^n) = \|D1_O\| < +\infty.$$

Let $s \in (0,1)$ be arbitrary.

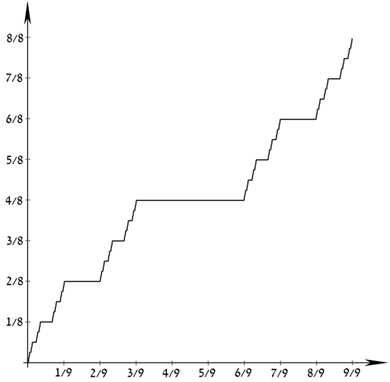
- If a smooth curve $X : [0,T] \to \mathbb{R}^n$, parametrized to have unit speed, hits ∂O in finitely many points then we have $X \in V(1_O, s, 1)$, but if it spends dt-positive time in ∂O then it cannot be an element of $V(1_O, s, 1)$.
- For $n = 1$ or $n = 2$ the path of a Bm is in $V(1_O, s, \infty)$ with probability one. For arbitrary $n \geqslant 1$, the path of a *fractional Brownian motion* (fBm) with *Hurst index* $H \in (0, \frac{1}{n-1+s})$ is in $V(1_O, s, \infty)$ with probability one.
- For arbitrary $n \geqslant 1$ it also follows that if $H \in (0, \frac{1}{s})$ and the fBm is started in $(\partial O)^c$ then it is in $V(1_O, s, 1)$ with probability one.

Cantor staircase

Let $\mathcal{C} \subset [0,1]$ be the classical middle third Cantor set and $\nu_{\mathcal{C}}$ the unique self-similar probability measure with support $\mathcal{C}$.

Let $\varphi : \mathbb{R}^n \to [0,1]$ be a function that is Lipschitz on $[0,1]^c \times \mathbb{R}^{n-1}$ and satisfies $\varphi(x) = \nu_{\mathcal{C}}((0,x_1))$ for all $x = (x_1, x_2, \ldots, x_n) \in [0,1] \times \mathbb{R}^{n-1}$.

Then $\varphi \in BV_{\text{loc}}(\mathbb{R}^n)$, and on $[0,1]^n$ we have $\|D\varphi\| = D\varphi = \nu_{\mathcal{C}} \otimes \mathcal{H}^{n-1}$.



Cantor staircase

- Writing $d_{\mathcal{C}} = \frac{\log 2}{\log 3}$ for the Hausdorff dimension of $\mathcal{C}$, we find that for $s \in (0, d_{\mathcal{C}})$ any path X in $\mathbb{R}^n$ is in $V(\varphi, s, \infty)$.
- Now suppose $s \in (d_{\mathcal{C}}, 1)$. The constant path $X \equiv (\frac{1}{2}, 0, \dots, 0)$ in $\mathbb{R}^n$ is in $V(\varphi, s, \infty)$, but the constant path $X \equiv (0, 0, \dots, 0)$ is not in $V(s, 1, 1)$.
- For $n = 1$ any smooth function $X : (0, T) \to (0, 1)$ with a finite number of critical points is in $V(\varphi, s, \infty)$.
- For $n = 2$ a smooth curve $X : (0, T) \to (0, 1)^2$, parametrized to have unit speed, does not have to be in $V(\varphi, s, 1)$. On the other hand a path of Bm is in $V(\varphi, s, \infty)$ with probability one.
- For $n \geqslant 3$ paths of the fBm with Hurst index $H \in (0, \frac{1}{n-1+s})$ are in $V(\varphi, s, \infty)$ with probability one.

Compositions

Lemma

Let $\varphi \in BV_{\mathrm{loc}}(\mathbb{R}^n)$ *and* $X \in V(\varphi, s, 1)$ *for some* $s \in (0,1)$. *Then for any pair of Lebesgue representatives* $\widetilde{\varphi}^{(1)}$ *and* $\widetilde{\varphi}^{(2)}$ *of* φ *we have*

$$\widetilde{\varphi}^{(1)}(X_t) = \widetilde{\varphi}^{(2)}(X_t)$$

at dt*-a.e.* $t \in [0, T]$.

Definition

Let $\varphi \in BV_{\mathrm{loc}}(\mathbb{R}^n)$ and suppose that $X \in V(\varphi, s, 1)$ for some $s \in (0,1)$. We define the *composition*

$$\varphi \circ X$$

to be the Lebesgue equivalence class of $t \mapsto \widetilde{\varphi}(X_t)$ on $[0, T]$, where $\widetilde{\varphi}$ is a Lebesgue representative of φ.