Stochastic pressure equation in enhanced geothermal heating

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joint work with

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SPDEs below sea level

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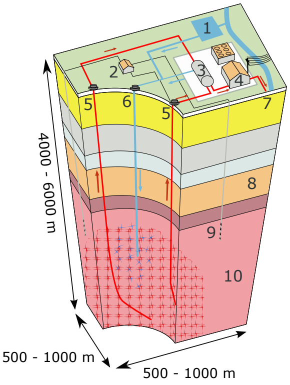
Contents

Section 1

Heuristics

Enhanced geothermal heating



- In the red area the earth is roughly 400°C.
- The red area is crystalline bedrock (rocks glued together with fossilized minerals).
- The rock can be made permeable by a process called hydroshearing.
- Goal: push water through the crystalline rock (6) and extract the heated water through (5), generate electricity and/or heat (7).
- Leads to *creeping water-flow* acting like a flow in porous media, however, different from "homogenized flow" due to the formation of "fluid channels".

Creeping flow and the pressure equation

Creeping water flow in porous media satisfies Darcy's Law, i.e.

$$\mathrm{v}(x) = -\frac{\kappa(x)}{\mu}(\nabla P - \mathrm{F})$$

κ is the permeability, μ dynamic viscosity of water (κ/μ is the diffusion coefficient),

v is the flux, P is the pressure and F is the underlying source field.

In the steady state, we get by conservation of mass that v is solenoidal, i.e.

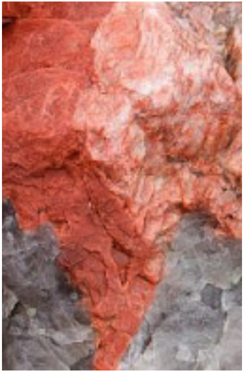
$$-\nabla \cdot \left(\frac{\kappa(x)}{\mu}\nabla P\right) = -\nabla \cdot \left(\frac{\kappa(x)}{\mu}\mathrm{F}\right) =: f$$

The pressure equation is the stationary equation ($\partial P/\partial t = 0$) of Darcy's fluid flow,

$$\frac{\partial P}{\partial t} - \nabla \cdot \left(\frac{\kappa(x)}{\mu}\nabla P(x,t)\right) = f.$$

Properties of crystalline rock

- The porosity is a measurement of how "solid" the rock is and can be measured using a method called neutron porosity.
- Porosity seems to have a so-called "power law" spectra ($1/f$, pink noise).
- This corresponds to correlation being logarithmic.
- It is conjectured that permeability κ is essentially the exponential of porosity.



The porosity field $\phi(x)$

It was first noted by [Hewett et al. (1986)] that the spatial correlation of crustal crack distributions follow a power-law $1/k$-scaling.

Considering the inverse problem in geothermal engineering leads to certain evidence [Leary et al. (2020)] that the spatial distribution of the porosity field $\phi(x)$ can be modeled by a *fractional Gaussian field** $\mathrm{FGF}_s(\mathbb{R}^d)$ with index $s = \frac{3}{2}$ in $d = 3$.

As a consequence, the permeability

$$\kappa(x) = e^{\alpha\mu + \beta\phi(x)}$$

where $\mu > 0$ is the *mean porosity*, and $\alpha, \beta > 0$ are suitably chosen parameters.

We choose zero mean $\mu = 0$. The choice of β reflects the size of fluctuations of porosity.

*Discussed on the following slides.

Fractional Gaussian fields

A *fractional Gaussian field* $\mathrm{FGF}_s(\mathbb{R}^d)$ with *index* $s \in \mathbb{R}$ is given by

$$(-\Delta)^{-s/2} W \sim \mathrm{FGF}_s(\mathbb{R}^d),$$

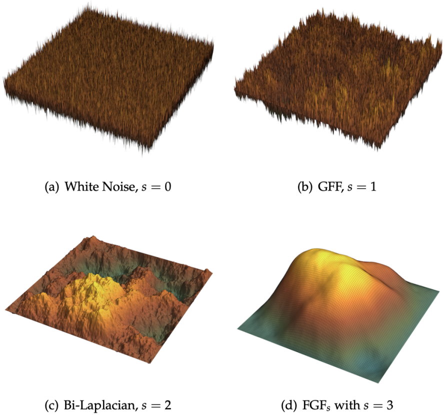
where W is *spatial white noise* in $\mathbb{R}^d$.

The scaling property of the law of $\mathrm{FGF}_s(\mathbb{R}^d)$ is captured by its *Hurst exponent*

$$H := s - \frac{d}{2}$$

such that, if $h \sim \mathrm{FGF}_s(\mathbb{R}^d)$, then for every $a > 0$, the field $x \mapsto h(ax)$ has the same law as $x \mapsto a^H h(x)$.

Fractional Gaussian fields



Surface plots of discrete fractional Gaussian fields on $[0,1]^2 \subset \mathbb{R}^2$ with zero boundary conditions.

GFF denotes the *Gaussian free field*.

Picture from [Lodhia, Sheffield, Sun, Watson (2016)].

Section 2

Stochastic pressure equation

Permeability and renormalization

Thus we shall assume the permeability is the exponential of porosity, i.e. the exponential of something log-correlated. A naïve model is

$$\kappa(x) = e^{\beta X}$$

for a parameter β and where X is a log-correlated Gaussian field on $U \subset \mathbb{R}^d$, i.e. X is a centered Gaussian field with

$$\mathbb{E}[X(x)X(y)] = -\log|x-y| + g(x,y), \quad x, y \in U,$$

for some $g \in C(U \times U)$.

For $d = 2$, X is the Gaussian free field, and

$$\mathbb{E}[X(x)X(y)] = G_U(x,y), \quad x, y \in U,$$

where G_U is the Green kernel of the Dirichlet Laplace operator $-\Delta$ on U.

However, we only have that a.s. $X \in W^{-\delta,2}(\mathbb{R}^d)$, $\delta > 0$, see [Junnila, Saksman, Webb (2019)]. Hence, we need to renormalize the field.

Gaussian multiplicative chaos measure (GMC)

The *Gaussian multiplicative chaos measure* μ_β is "defined" as

$$\mu_\beta := e^{\diamond \beta X} := e^{\beta X - \frac{\beta^2}{2} \text{“}\infty\text{”}} = e^{\beta X - \frac{1}{2}\operatorname{Var}(\beta X)}.$$

By mollifying X_ε, we may define

$$\langle e^{\diamond \beta X}, f \rangle := \lim_{\varepsilon \searrow 0} \int_U f \underbrace{e^{\beta X_\varepsilon - \frac{\beta^2}{2}\operatorname{Var}(X_\varepsilon)}}_{=: e^{\diamond \beta X_\varepsilon}} \, dx, \quad f \in C_0^\infty(U),$$

which only works if $\beta < \sqrt{2d}$, which yields the existence and non-triviality of μ_β.

We have L^p-moments $\mathbb{E}[\mu_\beta(U)^p] < \infty$ for all $p < \frac{2d}{\beta^2}$, and hence second moments for $\beta < \sqrt{d}$.

The pressure equation and more renormalizations

Consider the naïve equation

$$-\nabla \cdot \left(e^{\diamond \beta X} \nabla P \right) = f.$$

This does not necessarily make sense. The following, however, makes sense

$$-\nabla \cdot \left(e^{\diamond \beta X_\varepsilon} \nabla P \right) = f,$$

but P does not depend on f in the limit.

The associated parabolic problem ($f \equiv 0$) is supercritical

$$\frac{\partial P}{\partial t} = \nabla \cdot \left(e^{\diamond \beta X} \nabla P \right) \quad \text{“="} \quad e^{\diamond \beta X} \left[\Delta P + \nabla(\diamond \beta X) \cdot \nabla P \right].$$

Renormalization

We can now either renormalize the entire operator (LHS) or we can renormalize the product inside the divergence. The general issue that there is no canonical way to define the product of two distributions, is tackled by “renormalizing” the equation.

The Wick stochastic pressure equation on $\mathbb{T}^d$

Let $U = \mathbb{T}^d$ and $0 \leqslant \beta < \sqrt{d}$.

$$-\nabla \cdot \left(e^{\diamond(-\beta X)} \diamond \nabla P\right) = f,$$

where $\diamond$ is the so-called *Wick product*.

- The Wick product is a renormalized product of random variables, that can be extended to stochastic distributions (Hida, Kondratiev).
- The Wick product forms a ring on *Hida distributions* (commutative, etc.)
- The Wick product multiplies random variables as if they where independent $\mathbb{E}[X \diamond Y] = \mathbb{E}[X]\mathbb{E}[Y]$.
- If W_t is white noise then the *Skorohod integral* can be written as

$$\int_0^t X_s \delta B_s = \int_0^t X_s \diamond W_s ds$$

Renormalizing the product

Let $U = \mathbb{T}^d$ and $0 \leqslant \beta < \sqrt{d}$.

$$-\nabla \cdot \left(e^{\diamond(-\beta X)} \diamond \nabla P \right) = f,$$

where $\diamond$ is the so-called *Wick product*.

- The main benefit of the Wick product is that it preserves the expectation in the sense that $\overline{P} = \mathbb{E}[P]$ solves

  $$-\nabla \cdot \left(\mathbb{E}[e^{\diamond(-\beta X)}] \nabla \overline{P} \right) = \mathbb{E}[f]$$

  Thus we can think that P randomly deviates from $\overline{P}$, which is the solution of the averaged equation.
- When X is white noise, the above equation was studied by Holden, Øksendal, Ubøe, and Zhang (1993) to study the “same” problem.

The S-transform

The property $\mathbb{E}[X \diamond Y] = \mathbb{E}[X]\mathbb{E}[Y]$ is a special case of the following identity:

The S-transform

On a Gaussian Hilbert space $\mathcal{H}$ we define the *S-transform* for $X \in L^2_{\mathcal{H}}$, $SX : \mathcal{H} \to \mathbb{R}$ by

$$(SX)(h) = \mathbb{E}[Xe^{\diamond h}], \quad h \in \mathcal{H}.$$

The S-transform also satisfies the product property, i.e.,

$$(S[X \diamond Y])(h) = (SX)(h)(SY)(h),$$

which resembles the well-known corresponding identity for the Fourier transform.

The S-transform

The S transform thus transforms for each $h \in \mathcal{H}$

$$-\nabla \cdot \left(e^{\diamond \beta X} \diamond \nabla P \right) = f,$$

to

$$-\nabla \cdot \left(e^{-\beta \mathbb{E}[hX(\cdot)]} \nabla \overline{P}_h \right) = \overline{f}_h,$$

where $\overline{P}_h := (SP)(h)$, $\overline{f}_h := (SP)(h)$. Testing with $h = \beta X(z)$ for $0 < \beta < \sqrt{d}$, we get, at least in theory,

$$-\nabla \cdot \left(e^{-\beta^2 \mathbb{E}[X(z)X(x)]} \nabla \mathbb{E} \left[P(x) e^{\diamond X(z)} \right] \right) = f(x).$$

Heuristic derivation of a deterministic equation

$$-\nabla \cdot \left(e^{-\beta^2 \mathbb{E}[X(z)X(x)]} \nabla \mathbb{E}[P(x) e^{\diamond X(z)}] \right) = f(x).$$

Denote $u_z(x) = \mathbb{E}[P(x)e^{\diamond X(z)}]$, and note that $e^{-\beta^2 \mathbb{E}[X(z)X(x)]} = |x-z|^{\beta^2}$, thus we get the family of deterministic problems

$$-\nabla \cdot \left(|x-z|^{\beta^2} \nabla u_z(x) \right) = f(x).$$

Proposition. (Avelin, Kuusi, Nummi, Saksman, T., Viitasaari, arXiv:2405.17195)

The mapping from the stochastic side to the deterministic side is “invertible” if $z \mapsto u_z$ is in $W^{s_\beta,2}(U)$, $s_\beta = \frac{d-\beta^2}{2}$.

Regularity estimates

The equation

$$-\nabla \cdot \left(|x - z|^{\beta^2} \nabla u_z(x) \right) = f(x),$$

is a weighted elliptic equation where the weight is A_2-Muckenhoupt.

- From the works of Fabes, Jerison, Kenig, and Serapioni (80's) we know a lot about the regularity of said equation, but w.r.t x.
- Instead, we are interested in regularity w.r.t. z.
- The behavior of the solution when z is close to the boundary is delicate, and we cannot handle it at the moment.

Section 3

Main result

Solution in 1D

Theorem (Avelin, Kuusi, Nummi, Saksman, T., Viitasaari, arXiv:2402.09127)

$$-\nabla \cdot \left(e^{\diamond(-\beta X)} \diamond \nabla P\right) = f,$$

Neumann boundary conditions:

$$\begin{cases} e^{\diamond(-\beta X(0))} \diamond U'(0) = U_0 \\ e^{\diamond(-\beta X(T))} \diamond U'(T) = U_T \end{cases}$$

where U_0, U_T are deterministic constants, then the unique solution is given by

$$U(t) = \int_0^t (U_0 + \int_0^s f(s)ds)e^{\diamond \beta X} ds, \quad t \in (0, T).$$

Regularity result

Theorem (Avelin, Kuusi, Nummi, Saksman, T., Viitasaari, arXiv:2405.17195)

Let $w = |x-z|^{\beta^2}$, $\beta^2 < d$ *and let* $u \in H^1(\mathbb{T}^d; w)$ *be a weak solution to the equation*

$$\begin{cases} -\nabla \cdot (|x-z|^{\beta^2} \nabla u(\cdot; z)) = \nabla \cdot (|x-z|^{\beta^2} \mathrm{f}(\cdot)), & \text{in } \mathbb{T}^d, \\ \int_{\mathbb{T}^d} u(x; z) dx = 0. \end{cases}$$

There exists a $\varepsilon = \varepsilon(\beta, d)$ *and such that if* $\mathrm{f} \in W^{d/2,q}(\mathbb{T}^d; w)$ *for* $q > 2d - \varepsilon$, *then there exists a* $\gamma = \gamma(\beta, d, q) \in (0,1)$ *and* $C = C(\beta, d, q) \geqslant 1$ *such that for any multi-index* $|\alpha| \leqslant d/2$ *we have for* $x \neq z$,

$$|\partial_z^\alpha u(x; z)| \leqslant C|x-z|^{\gamma - |\alpha|} \|\mathrm{f}\|_{W^{d/2,q}(\mathbb{T}^d; w)}.$$

Thus, uniformly in x, *it holds*

$$u(x; \cdot) \in W^{d/2,2}(\mathbb{T}^d).$$

Main existence and uniqueness result

Theorem (Avelin, Kuusi, Nummi, Saksman, T., Viitasaari, arXiv:2405.17195)

The stochastic Wick pressure equation with log-correlated noise on $\mathbb{T}^d$ admits a solution P for data $\mathrm{f} \in W^{d/2,q}(\mathbb{T}^d; w)$, $q \geqslant 2d$, $w = |x-z|^{\beta^2}$, and $0 \leqslant \beta < \sqrt{d}$. The solution has a representation

$$P(x) = \int_{\mathbb{T}^d} \varphi(x,y)\, \mu_\beta(dy),$$

where $\varphi(\cdot, y) \in W^{1,2}(\mathbb{T}^d; w)$ and $\varphi(x, \cdot) \in W^{-s_\beta,2}(\mathbb{T}^d)$, $s_\beta = \frac{d-\beta^2}{2}$. The solution is unique in the class of solutions with this representation.

Wrap-up

Stochastic pressure equation

- Motivated by fluid flow in crustal rock from enhanced geothermal heating
- Data from geophysical engineering suggest that permeablility can be modeled as exponential of log-correlated Gaussian field
- Terms in the pressure equation have to be renormalized due as the product of random distributions fails to exist otherwise
- Stochastic equation is S-transformed to family of deterministic equations depending on a parameter
- Regularity theory guarantees invertiblity of the S-transform
- Stochastic equation has solution that is unique in the class of solutions "with an S-transform"

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Thank you for your attention!

Section 4

Backup

Regularity result

Theorem (Avelin, Kuusi, Nummi, Saksman, T., Viitasaari, arXiv:2405.17195)

Let $w = |x - z|^{\beta^2}$, $\beta^2 < d$ and let $u \in H^1(\mathbb{T}^d; w)$ be a weak solution to the equation

$$\begin{cases} -\nabla \cdot (|x - z|^{\beta^2} \nabla u(\cdot; z)) = \nabla \cdot (|x - z|^{\beta^2} \mathrm{f}(\cdot)), & \text{in } \mathbb{T}^d, \\ \int_{\mathbb{T}^d} u(x; z) dx = 0. \end{cases}$$

There exists a $\varepsilon = \varepsilon(\beta, d)$ and such that if $\mathrm{f} \in W^{d/2,q}(\mathbb{T}^d; w)$ for $q > 2d - \varepsilon$, then there exists a $\gamma = \gamma(\beta, d, q) \in (0, 1)$ and $C = C(\beta, d, q) \geqslant 1$ such that for any multi-index $|\alpha| \leqslant d/2$ we have for $x \neq z$,

$$|\partial_z^\alpha u(x; z)| \leqslant C|x - z|^{\gamma - |\alpha|} \|\mathrm{f}\|_{W^{d/2,q}(\mathbb{T}^d; w)}.$$

Thus, uniformly in x, it holds

$$u(x; \cdot) \in W^{d/2,2}(\mathbb{T}^d).$$

Fix $x^*, z^* \in \mathbb{T}^d$ such that $x^* \neq z^*$, and let $r = |x^* - z^*|/16$.

- Let $\Phi(x) = x + \eta(x - z^*)h$, where $\chi_{B_r} \leqslant \eta \leqslant \chi_{B_{2r}}$.
- Establish the expansion

$$u(\Phi(\cdot); z^*) - u(\Phi(\cdot); z^* + h) = \sum_{1 \leqslant |\alpha| \leqslant d/2} v^{(\alpha)} h^\alpha + o(|h|^{d/2})$$

- Each coefficient satisfies

$$-\nabla \cdot \left(w \nabla v^{(\alpha)} \right) = \nabla \cdot (w \mathcal{E}^{(\alpha)} \nabla u) + \sum_{\substack{1 \leqslant |\alpha_2|, |\alpha_1| \leqslant m, \\ \alpha_1 + \alpha_2 = \alpha}} \nabla \cdot \left(w \mathcal{E}^{(\alpha_2)} \nabla v^{(\alpha_1)} \right) + \nabla \cdot (w \mathcal{F}^{(\alpha)})$$

- Use weighted elliptic regularity estimates to obtain local boundedness of $v^{(\alpha)}$

$$|v^{(\alpha)}(x^*)| \lesssim C(|\alpha|) r^{\gamma - |\alpha|} \|f\|_{W^{|\alpha|, q}(\mathbb{T}^d; w)}.$$