# Quantitative mixing for locally monotone stochastic evolution equations

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**SPDEs: theory and applications**

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# 1 Motivation

Consider the stochastic incompressible 2D Navier-Stokes equations (NSE) with additive Wiener noise on a Lipschitz domain of finite width[1] $\mathcal{O}\subset\mathbb{R}^2$, viscosity $\nu>0$,

$$dX_t=[\nu\Delta X_t+(X_t\cdot\nabla)X_t]\,dt+B\,dW_t, \qquad t>0,$$

$$\nabla\cdot X_t=0, \qquad t\geq 0,$$

$$X_0=x\in L^2_{\mathrm{sol}}(\mathcal{O};\mathbb{R}^2)=:H\,,$$

with no-slip boundary condition $X_t=0$ on $\partial\mathcal{O}$. Alternatively, $\mathcal{O}=\mathbb{T}^2$, $\int_{\mathbb{T}^2}X_t\,dz=0$

By the spectral Galerkin method, if $B\in\mathrm{HS}(U,H)$ and if $\{W_t\}_{t\geq 0}$ is a cylindrical Wiener noise modelled on a separable Hilbert space $U$, there exists a unique adapted Markovian strong solution

$X^x\in L^2([0,T]\times\Omega;H^1_{\mathrm{sol},0}(\mathcal{O};\mathbb{R}^2))$.

1. That is, $\mathcal{O}$ fits between two parallel lines. Possibly unbounded, but satisfies a Poincaré inequality.

# Locally monotone drift SPDEs



The 2D NSE is an example of a *locally monotone drift SPDE*.

$$dX_t = A(X_t)\,dt + B\,dW_t, \qquad t \geq 0.$$

Let $V \subset H \equiv H^* \subset V^*$ be a Gelfand triple.

$V$ separable, reflexive Banach space, $H$ separable Hilbert space.

$A\colon V \to V^*$ is called *monotone* if there exists $K \in \mathbb{R}$,

$$\langle A(u) - A(v), u - v \rangle \leq K \|u - v\|_H^2, \qquad u, v \in V.$$

If $H = V = \mathbb{R}$, $f\colon \mathbb{R} \to \mathbb{R}$ is monotone **iff** $x \mapsto f(x) - Kx$ is non-increasing.

$A$ is called *locally monotone* in $V$ if there exists $\rho\colon V \to \mathbb{R}$, locally bounded and measurable, and $K \in \mathbb{R}$, such that

$$\langle A(u) - A(v), u - v \rangle \leq (K + \rho(u)) \|u - v\|_H^2, \qquad u, v \in H.$$

# Further examples



**Further examples of locally monotone drift SPDEs**

- Stochastic power-law fluid equations
- Stochastic Allen-Cahn equation
- Stochastic Burgers equation

**Non-monotone perturbations of monotone drift SPDEs**

- Stochastic $p$-Laplace equation $+$ perturbation
- Stochastic porous medium-type equations $+$ perturbation

**(not covered by our ergodicity result)**

## Further examples of locally monotone drift SPDEs

- Stochastic power-law fluid equations on a domain of finite width $\mathcal{O} \subset \mathbb{R}^d$, $p \in (1, \infty)$,

$$dX_t = [\nabla \cdot S(\boldsymbol{e}(X_t)) + (X_t \cdot \nabla) X_t]\, dt + B\, dW_t, \qquad t > 0,$$

$$\nabla \cdot X_t = 0, \qquad t \geq 0,$$

$$X_0 = x \in L^2_{\mathrm{sol}}(\mathcal{O}; \mathbb{R}^d),$$

where

$$\boldsymbol{e}(u)_{i,j} := \frac{1}{2}(\partial_i u_j + \partial_j u_i),$$

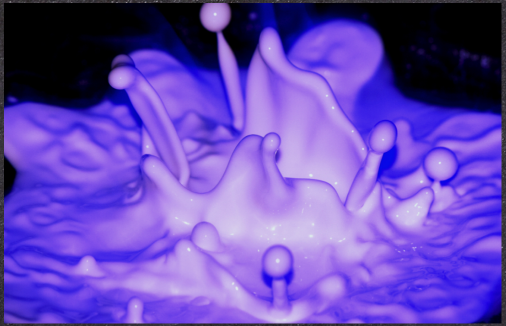
and

$$S(z) := 2\nu(1 + |z|)^{p-2} z.$$

## Further examples of locally monotone drift SPDEs

- Stochastic power-law fluid equations on $\mathcal{O} \subset \mathbb{R}^d$, $p \in (1, \infty)$, $d = 2, 3$.

| $p > 2$ | $p < 2$ | $p = 2$ |
|---|---|---|
| *dilatant*, or *shear-thickening* | *pseudoplastic*, or *shear-thinning* | *Newtonian*, Navier-Stokes equations |
| oobleck | hair gel, blood, whipped cream | Examples: water, glycerol, ethanol |
| (mixture of water and corn-starch) | (polymeric molecules) | viscous stress $\propto$ local strain rate |



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**Further examples of locally monotone drift SPDEs**

- Stochastic Allen-Cahn equation on $\mathcal{O} \subset \mathbb{R}^d$, $d = 1, 2, 3$.

$$dX_t = [\nu \Delta X_t + g(X_t)]\, dt + B\, dW_t, \qquad t > 0,$$

$$X_0 = x \in L^2(\mathcal{O}).$$

Typically, $g(z) = z - z^3$.

- Stochastic Burgers equation on $\mathcal{O} \subset \mathbb{R}^d$, $d = 1, 2, 3$.

$$dX_t = [\nu \Delta X_t + \langle \boldsymbol{f}(X_t), \nabla X_t \rangle]\, dt + B\, dW_t, \qquad t > 0,$$

$$X_0 = x \in L^2(\mathcal{O}).$$

1D stochastic Burgers, $d = 1$, $\boldsymbol{f}(z) = z$.

# Main question



Existence and uniqueness of solutions to the examples for any finite time horizon $T > 0$ have been discussed in Liu, Röckner [LR10].

**What about $T \to \infty$?**

Replacing the noise $B\,dW$ by a deterministic forcing $f \in L^2_{\mathrm{sol}}$, the 2D NSE with no-slip boundary condition is known to have **exponential convergence** to the stationary solution [Tem01], when **the viscosity is large enough** relative to constants depending only on the **domain**, the **first eigenvalue** of the Stokes operator, and the $L^2_{\mathrm{sol}}$-norm of the **deterministic forcing** $f$.

Without forcing, the solution has **exponential decay** to zero.

**What about the stochastic case?**

# 2 Invariant measures

Set $P_tF(x) := \mathbb{E}[F(X_t^x)]$, $t \geq 0$, $F \in \mathcal{B}_b(H)$, $x \in H$ and define $P_t^*$ by duality

$$\langle P_t^*\mu, F\rangle = \langle P_tF, \mu\rangle, \quad t \geq 0,\ \mu \in \mathcal{M}_1(H),\ F \in C_b(H).$$

**Definition 1.** *A probability measure $\mu \in \mathcal{M}_1(H)$ is said to be* invariant *for $\{P_t\}$ if $P_t^*\mu = \mu$ for every $t \geq 0$.*

*$\{P_t\}_{t\geq 0}$ is called* Feller *if $P_t(C_b(H)) \subset C_b(H)$ for every $t \geq 0$.*

*$\{P_t\}$ is called weak$^*$ mean ergodic if for some invariant measure $\mu$,*

$$\frac{1}{T}\int_0^T P_t^*\nu\, dt \rightharpoonup \mu, \quad \text{as} \quad T \to \infty$$

*for every $\nu \in \mathcal{M}_1(H)$ which is equivalent to the uniqueness of $\mu$.*

# Krylov-Bogoliubov theorem



**Proposition 2.** *(Krylov-Bogoliubov).*

*If for a Feller semigroup* $\{P_t\}$ *and some* $x \in H$, $t_n \nearrow \infty$, $\mu \in \mathcal{M}_1(H)$,

$$\frac{1}{t_n}\int_0^{t_n} \mathrm{Law}(X_s^x)\,ds \rightharpoonup \mu \quad \text{as} \quad n \to \infty$$

*then* $\mu$ *is an invariant measure for* $\{P_t\}$.

**Remark 3.** This method can be used to prove that the stochastic 2D NSE with additive Gaussian forcing admits an invariant measure by standard a priori estimates.

**Definition 4.** *A Feller semigroup* $\{P_t\}$ *is said to have the* strong Feller property *if*

$$P_t(\mathcal{B}_b(H)) \subset C_b(H) \quad \text{for every} \quad t > 0.$$

A Feller semigroup $\{P_t\}$ is said to be *irreducible* if for every $t > 0$ and for every $x \in H$ and for every non-empty open set $O \subset H$,

$$P_t \mathbb{1}_O(x) > 0.$$

**Theorem 5.** *If* $\{P_t\}$ *is strong Feller and irreducible, it admits a unique invariant measure* $\mu$ *such that for every* $x \in H$

$$||P_t^* \delta_x - \mu||_{\mathrm{TV}} \to 0 \quad \text{as} \quad t \to \infty.$$

# Timeline (some highlights)



- In 1995, Flandoli and Maslowski [FM95] proved the strong Feller property and irreducibility of the stochastic 2D NSE with non-degenerate Gaussian noise.
- In 2001–2002, the (exponential) ergodicity of the stochastic 2D NSE with either non-degenerate forcing or large viscosity $\nu$ was proved by Bricmont, Kupiainen, and Lefevere [BKL02]; E, Mattingly, and Sinai [EMS01]; Kuksin and Shirikyan [KS01]; Mattingly [Mat02]; ...
- In 2002, the general well-posedness theory for the stochastic 2D NSE was discussed by Menaldi and Sritharan [MS02].
- In 2006, Hairer and Mattingly [HM06] provided a minimal non-degeneracy condition of the noise for the *asymptotic strong Feller property* and *weak irreducibility* with a minimal number of four forced modes.
- **And many others ...**

  Recently, there has been a lot of progess for multiplicative noise, Lévy noise, pure jump noise, and coupling techniques.

# Mild degeneracy



Denote by $P_n$ the Galerkin projection on the first $n$ Fourier modes. If the noise coefficient $B$ is *mildly degenerate*, that is, $B \in \mathrm{HS}(H)$ and for every $\nu > 0$, there exists $N = N(\nu, \|B\|_{\mathrm{HS}(H)})$ such that, if

$$\mathrm{Rg}(B) \supset P_n(H),$$

for some $n \geq N$, then the stochastic 2D NSE admits a unique invariant measure, and the *Foias-Prodi estimate* holds in the stochastic case for some $C = C(x, y, B, \mathcal{O}, \nu) > 0$, and $\delta = \delta(B, \mathcal{O}, \nu) > 0$ such that

$$\mathbb{E}[\|X_t^x - X_t^y\|_H^2] \leq Ce^{-\delta t}$$

see Glatt-Holtz, Mattingly, Richards [GMR17]. Thus, we get exponential mixing. The estimates rely on the properties of the **exponential martingale**.

# The $e$-property

**Definition 6.** *A Feller semigroup* $\{P_t\}$ *is said to satisfy the* $e$-property *if for every* $\varphi \in \mathrm{Lip}_b(H)$*, for every* $x \in H$*, for every* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that*

$$|P_t\varphi(x) - P_t\varphi(y)| < \varepsilon$$

for every $t \geq 0$ and for every $y \in H$ with $\|x - y\| < \delta$.

This type of uniform equicontinuity for bounded Lipschitz functions could be viewed a **coupling condition at infinity**.

It has been conjectured by Szarek and Worm [SW12] that

> "It seems that all known examples of Markov processes with the asymptotic strong Feller property satisfy the $e$-property as well."

Jaroszewska constructed a counter-example in 2013 in an unpublished preprint.

# The "lower bound technique"

Originally developed by Lasota and Szarek [LS06], the "lower bound technique" can be described as follows in the work of Komorowski, Peszat and Szarek [KPS10].

**Theorem 7.** *Assume that* $\{P_t\}$ *is Feller and has the* $e$*-property. Assume that there exists* $z \in H$ *such that for every bounded set* $J \subset H$ *and every* $\delta > 0$

$$\inf_{x\in J}\ \liminf_{T\to\infty}\frac{1}{T}\int_0^T P_t \mathbb{1}_{B(z,\delta)}(x)dt > 0.$$

*Suppose further that for every* $\varepsilon > 0$ *and every* $x \in H$ *there exists a bounded Borel set* $K \subset H$ *such that*

$$\liminf_{T\to\infty}\frac{1}{T}\int_0^T P_t \mathbb{1}_K(x)dt > 1-\varepsilon.$$

*Then there exists a unique invariant probability measure* $\mu$ *for* $\{P_t\}$*, and* $\{P_t\}$ *is weak*$^*$ *mean ergodic.*

The following conditions (more generally for couplings) have been inspired by Butkovsky, Kulik and Scheutzow [BKS20] (adapted to our situation). For a lower semi-continuous function $U\colon H \to [0,\infty)$ and a measurable function $S\colon H \to [0,\infty]$,

$$\|X_t^x - X_t^y\|_H^2 \leq \|x-y\|_H^2 \exp\left(-\zeta t + \kappa\int_0^t S(X_s^x)\,ds\right), \quad t \geq 0,$$

$$U(X_t^x) + \mu\int_0^t S(X_s^x)\,ds \leq U(x) + bt + M_t, \quad t \geq 0,$$

where $\mu > 0$ and $b \geq 0$, such that

$$\zeta > \frac{\kappa b}{\mu},$$

and $M$ is a continuous local martingale with $M_0 = 0$ and for $b_1, b_2 \geq 0$,

$$d\langle M\rangle_t \leq b_1 S(X_t^x)\,dt + b_2\,dt, \quad t \geq 0.$$

# 3 Main result

Consider

$$(1) \quad dX_t = A(X_t)\,dt + B\,dW_t, \qquad t > 0, \qquad X_0 = x_0,$$

where $\{BW_t\}$ is as before, i.e., $\{W_t\}$ is a **cylindrical Wiener process** on a separable Hilbert space $U$ and $B \in \mathrm{HS}(U, H)$.

The drift $A\colon V\to V^*$ is *hemicontinuous* (i.e., weakly continuous along rays) and for $\alpha\geq 2$, $\delta_1>0$, $u\in V$,

$$2\langle A(u),u\rangle \leq -\delta_1\|u\|_V^\alpha \quad \text{(coercivity)}$$

and for $\delta_2>0$, $C_2\geq 0$, $u,v\in V$,

$$2\langle A(u)-A(v),u-v\rangle \leq (-\delta_2+\rho(u))\|u-v\|_H^2, \quad \text{(local monotonicity)}$$

where for $\beta\geq 0$

$$0\leq \rho(u)\leq C_2\|u\|_V^\alpha\|u\|_H^\beta \quad u\in V.$$

Moreover, for $K>0$, $u\in V$,

$$\|A(u)\|_{V^*}^{\frac{\alpha}{\alpha-1}}\leq K(1+\|u\|_V^\alpha)(1+\|u\|_H^\beta) \quad \text{(boundedness)}.$$

Liu and Röckner (for Wiener noise) [LR10] and Brzezniak, Liu, and Zhu (for Lévy noise) [BLZ14] proved the following.

**Theorem 8.** *Under the previous hypotheses, for every initial datum*

$$x_0 \in L^{\beta+2}(\Omega, \mathcal{F}_0, \mathbb{P}; H)$$

*there exists a unique continuous strong adapted solution* $\{X_t\}$ *with*

$$X \in L^{\alpha}([0,T]; V) \cap L^2([0,T]; H) \quad \mathbb{P}\text{-}a.s.$$

*such that every progressively measurable* $V$*-valued version of* $X$ *satisfies (1)* $\mathbb{P}$*-a.s.*

**Note that** our assumptions are intentionally not the most general ones, and exclude e.g. the stochastic $p$-Laplace equation, $p \neq 2$, or time-dependent drift.

# Non-standard hypothesis on the drift



Assume also that there exist $\delta_4 > 0$ and $C_4 \in \mathbb{R}$ such that for all $u \in V$,

$$2\langle A(u), u\rangle \leq C_4 - \delta_4 \|A(u)\|_{V^*} \quad \text{(cone condition)}.$$

**Remark 9.** This condition is satisfied for the 2D NSE and the power law fluid equations. Unfortunately, it is quite restrictive for *semilinear equations* with drift $A = A_0 + F$ because it forces the nonlinear perturbation $F$ of the dissipative principal term $A_0$ to have **at most quadratic growth**.

For this reason, our result does not cover stochastic Allen-Cahn equations with cubic nonlinearity.

# Quantitative conditions for ergodicity and mixing



Assume that the noise is spatially regular: $B \in \mathrm{HS}(U,V)$ (if $V$ is a Hilbert space, otherwise assume a small ball property in $V$).

To control the exponential martingale, we need to assume that

$$\alpha \geq 2, \quad 0 \leq \beta \leq \alpha - 2,$$

and a relation between $\|B\|^2_{\mathrm{HS}(U,H)} \geq 0$, $\delta_1, \delta_2 > 0$, $C_2 \geq 0$, and $c_0 > 0$ such that

$$\|v\|_V \geq c_0 \|v\|_H, \quad v \in V,$$

as follows: If $\beta = 0$, $\alpha = 2$, we assume that there exists $\gamma \in [0, \delta_2]$

$$\|B\|^2_{\mathrm{HS}(U,H)} \leq \frac{\delta_1}{C_2}(\delta_2 - \gamma) \wedge \frac{1}{4}\delta_1 c_0^2$$

If $\alpha > 2$, or $\beta > 0$, we assume an additional condition of similar type.

# The constants



For the **2D NSE**, we have that $\alpha = 2$, $\beta = 0$, $\delta_1 = 2\nu$, $\delta_2 = \nu c_0^2$, $C_2 = \frac{4}{\nu}$ (growth condition does not hold), or $\alpha = 2$, $\beta = 2$, $\delta_1 = 2\nu$, $\delta_2 = \nu c_0^2$, $C_2 = \frac{128}{\nu^3}$ (growth condition holds), see [BKS20] and [BLZ14]. Here, $c_0$ is the inverse of the Poincaré constant of the domain $\mathcal{O} \subset \mathbb{R}^2$. Note that in [BKS20], it is assumed also that there exists $n \in \mathbb{N}$ with

$$\mathrm{Rg}(B) \supset P_n(H),$$

**which we do not have to assume here**. We will use both above assumptions in different steps of the proof.

For the **1D Burgers equation**, we have that there exists $\varepsilon \in (0, 2\nu c_0^2)$, and $C = C(\varepsilon, c_0, \mathcal{O}) > 0$, such that:

$\alpha = 2$, $\beta = 0$, $\delta_1 = 2\nu$, $\delta_2 = 2\nu c_0^2 - \varepsilon$, $C_2 = C$.

Certainly, the **stochastic heat equation** is also covered with:

$\alpha = 2$, $\beta = 0$, $\delta_1 = 2\nu$, $\delta_2 = 2\nu c_0^2$, $C_2 = 0$.

**Theorem 10.** *(Barrera, T., 2025+) Under the previous hypotheses, the semigroup associated to (1) is Markovian and Feller, and satisfies the $e$-property and is weak* mean ergodic. The unique invariant probability measure $\mu$ on $(H, \mathcal{B}(H))$ admits finite $(\alpha+\beta)$-moments in $H$ and finite $\alpha$-moments in $V$.*

The existence and uniqueness of $\mu$ is proved by the $e$-property and the lower-bound technique.

Let $\tau^x(\varepsilon) := \inf\{t \geq 0 : \mathcal{W}_2(\mathrm{Law}(X_t^x), \mu) \leq \varepsilon\}$ be the $\varepsilon$-**mixing time** with prescribed error $\varepsilon > 0$, here $\mathcal{W}_2$ denotes the Wasserstein-2-distance.

**Theorem 11.** *(Barrera, T., 2025+) Under the previous hypotheses, if $\gamma > 0$, there exists a constant $K \geq 0$, such that we obtain the following upper bound:*

$$\tau^x(\varepsilon) \leq \frac{2}{\gamma}\left[\frac{C_2\|x\|_H^{\beta+2}}{4\delta_1(\beta+2)} + \log\left(\|x\|_H + \left(\frac{K}{2c_0^\alpha\delta_1(\beta+2)}\right)^{1/(\alpha+\beta)}\right) + \log\left(\frac{1}{\varepsilon}\right)\right].$$

**Theorem 12.** *(Barrera, T., 2025+) Assume that there exists* $\gamma \in (0, \nu c_0^2]$*, such that*

$$\|B\|^2_{\mathrm{HS}(U,H)} \leq \frac{1}{8}\nu^2(\nu c_0^2 - \gamma) \wedge \frac{1}{2}\nu c_0^2,$$

*then we obtain the following upper bound for the* $\varepsilon$*-mixing time*

$$\tau^x(\varepsilon) \leq \frac{2}{\gamma}\left[\frac{2\|x\|^2_H}{\nu^2} + \log\left(\|x\|_H + \frac{\|B\|_{\mathrm{HS}(U,H)}}{\sqrt{\nu}c_0}\right) + \log\left(\frac{1}{\varepsilon}\right)\right]$$

*for the stochastic 2D Navier-Stokes equations.*

# 4 Sketch of the proof

We first need to verify the $e$-property.

By a Galerkin approximation and Itô's lemma, we obtain the following pathwise a priori esimate for some constant $C>0$, not depending on $T>0$,

$$\|X_t^x\|_H^{\beta+2}+\delta_1\frac{\beta+2}{4}\int_0^T\|X_t\|_V^{\alpha}\|X_t\|_H^{\beta}dt$$

$$\leq\|x\|^{\beta+2}+CT+M_T,$$

where $t\mapsto M_t$ is a local martingale with $M_0=0$. The absolutely continuous terms on the RHS have been compensated with the 2nd term on the LHS by our hypotheses and Young's inequality.

For the 2D NSE, these types of estimates are well-known. In particular, applied for $\beta=0$, we obtain the existence of at least one invariant measure by the Krylov-Bogoliubov theorem whenever the the Sobolev embedding $V\subset H$ is compact.

**Recall:**

**Definition 13.** *A Feller semigroup* $\{P_t\}$ *is said to satisfy the* $e$-property *if for every* $\varphi \in \mathrm{Lip}_b(H)$, *for every* $x \in H$, *for every* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that*

$$|P_t\varphi(x) - P_t\varphi(y)| < \varepsilon$$

for every $t \geq 0$ and for every $y \in H$ with $\|x - y\| < \delta$.

$$|P_t\varphi(x) - P_t\varphi(y)| \leq \|\varphi\|^2_{\mathrm{Lip}}\,\mathbb{E}[\|X_t^x - X_t^y\|_H^2]$$

Note that if $A$ is monotone, we get that

$$\mathbb{E}[\|X_t^x - X_t^y\|_H^2] = \|x-y\|_H^2 + 2\mathbb{E}\int_0^t \langle A(X_s^x) - A(X_s^y), X_s^x - X_s^y\rangle ds \leq \|x-y\|_H^2.$$

# Gronwall's lemma



Note that because we have additive noise, for $0 < \gamma < \delta_2$, by local monotonicity,

$$\frac{d}{dt}\|X_t^x - X_t^y\|_H^2$$

$$\leq \|X_t^x - X_t^y\|_H^2(-\delta_2 + C_2\|X_t^x\|_V^\alpha\|X_t^x\|_H^\beta), \quad t > 0.$$

Hence by Gronwall's lemma,

$$\mathbb{E}\|X_T^x - X_T^y\|_H^2$$

$$\leq \|x - y\|_H^2\,\mathbb{E}\left[\exp\left(-\delta_2 T + C_2\int_0^T \|X_t^x\|_V^\alpha\|X_t^x\|_H^\beta dt\right)\right], \quad T \geq 0.$$

We are left with controlling the exponential moments, which is not obvious.

# The exponential martingale



By our assumption $\alpha \geq \beta - 2$, we can even obtain for another constant $\tilde{C} > 0$,

$$\|X_t^x\|_H^{\beta+2} + \delta_1 \frac{\beta+2}{4} \int_0^T \|X_t\|_V^{\alpha} \|X_t\|_H^{\beta} dt$$

$$\leq \|x\|^{\beta+2} + \tilde{C}T + M_T - \frac{1}{2}\langle M \rangle_T \,.$$

Now,

$$\mathbb{E}\left[\exp\left(M_T - \frac{1}{2}\langle M \rangle_T\right)\right] = 1.$$

Hence, for the $e$-property, we just need

$$\delta_2 \geq \frac{4C_2}{\delta_1(\beta+2)}\tilde{C},$$

which follows from our hypotheses.

Consider the deterministic counterpart to (1),

$$du_t^x = A(u_t^x)\,dt, \quad t > 0, \quad u_0^x = x,$$

and note that by coercivity, for every $R > 0$,

$$\lim_{t\to\infty} \sup_{\|x\|_H \le R} \|u_t^x\|_H = 0.$$

Let us prove that for any $T > 0$, for any $\varepsilon > 0$, and any $K \subset H$ bounded, we have

$$\mathbb{P}(\|X_T^x - u_T^x\|_H^2 < \varepsilon) > 0,$$

uniformly for $x \in K$.

# The “lower bound”



To prove this **stochastic stability** for finite times with positive probability, we use a pathwise argument to estimate $\|Y_t^x - u_t^x\|_H^2$, where $Y_t^x := X_t^x - BW_t$ is the solution to the random PDE

$$dY_t = A(Y_t + BW_t)\,dt, \quad t > 0, \quad Y_0 = x.$$

To control the error terms with positive probability, we need the *small ball property* of $\{BW_t\}$ in $V$, that is,

$$\mathbb{P}\left(\sup_{0 \le t \le T} \|BW_t\|_V < \delta\right) > 0.$$

which is automatically satisfied if $V$ is a Hilbert space and $B \in \mathrm{HS}(U, V)$. For the pathwise argument, we also need the non-standard assumption that there exist $\delta_4 > 0$ and $C_4 \in \mathbb{R}$ such that for all $u \in V$,

$$2\langle A(u), u\rangle \le C_4 - \delta_4 \|A(u)\|_{V^*}.$$

# The “lower bound” cont’d



We obtain

$$\mathbb{P}\Bigg(\sup_{0\le t\le T}\|X_t^x-u_t^x\|_H^2<\varepsilon\Bigg)$$

$$\geq\mathbb{P}\Bigg(\sup_{0\le t\le T}\|Y_t^x-u_t^x\|_H^2<\frac{\varepsilon}{4},\ \sup_{0\le t\le T}\|L_t\|_H^2<\frac{\varepsilon}{4}\Bigg)>0.$$

However, in the estimates for $\sup_{0\le t\le T}\|Y_t^x-u_t^x\|_H^2$, we in fact need the stronger requirement

$$\mathbb{P}\Bigg(\sup_{0\le t\le T}\|L_t\|_V^2<\delta\Bigg)>0,$$

and an application of the **stochastic Gronwall lemma** by Sarah Geiss [Gei24] (see also von Renesse and Scheutzow [vRS10]).

# The "lower bound" cont'd



Now, for every $\delta > 0$, and every $z \in K$, there exists $\gamma_1 > 0$, and $T_0 > 0$ such that

$$P_T \mathbb{1}_{B_\delta(0)}(z) = \mathbb{P}(\|X^z_{T_0}\|_H \leq \delta) \geq \mathbb{P}\left( \|X^z_{T_0} - u^z_{T_0}\|_H \leq \frac{\delta}{2}\right) \geq \gamma_1 > 0.$$

By a well-know trick used for instance by Es-Sarhir and von Renesse [EvR12], we may use the Markov property of the semigroup (shifting by $T_0$) to obtain

$$\liminf_{T\to\infty} \frac{1}{T}\int_0^T P_t \mathbb{1}_{B_\delta(0)}(x)\,dt \geq \liminf_{T\to\infty} \gamma_1 \frac{1}{T}\int_0^T P_t \mathbb{1}_K(x)\,dt > 0.$$

Here, we have also used that the coercivity implies for every $\varepsilon > 0$ and every bounded set $J$, there exists a bounded set $K$ with

$$\inf_{x\in J} \liminf_{T\to\infty} \frac{1}{T}\int_0^T P_t \mathbb{1}_K(x)\,dt > 1 - \varepsilon.$$

# Final remarks

Thus, the conditions of Komorowski, Peszat and Szarek [KPS10] can be verified.

This proves our main result. The moment estimates follow from our a priori estimate and the fact that $V \subset H$ is a continuous linear embedding.

**Theorem 14.** *(Barrera, T.) Under the previous hypotheses, the semigroup associated to (1) is Markovian and Feller, and satisfies the* $e$*-property and is weak*$^*$ *mean ergodic. The unique invariant probability measure* $\mu$ *on* $(H, \mathcal{B}(H))$ *admits finite* $(\alpha + \beta)$*-moments in* $H$ *and finite* $\alpha$*-moments in* $V$*.*

**Remark 15.**

- Due to our method, our proof is restricted to locally monotone equations, where $\rho$ depends **only on one variable** and not on both (i.e., *fully locally monotone*).
- The $V$**-regularity** of the noise and the **cone condition** are technical assumptions.

# 5 Stochastic $p$-Laplace equation

Let $p > 1$, and consider

$$dX_t^x = \operatorname{div}\left[|\nabla X_t^x|^{p-2}\nabla X_t^x\right]dt + B\,dW_t, \quad t > 0, \quad X_0^x = x,$$

with Dirichlet boundary conditions on a bounded domain $\mathcal{O} \subset \mathbb{R}^d$.

As a **toy example**, consider the SDE

$$dY_t^x = -|Y_t^x|^{p-2}Y_t^x\,dt + dW_t, \quad t > 0, \quad Y_0^x = x.$$

The generator has a spectral gap, and therefore the semigroup is **exponentially ergodic**. On the other hand, the ODE

$$y_t' = -|y_t|^{p-2}y_t, \quad t > 0, \quad y_0 = x,$$

has **polynomial decay** to zero for $p > 2$ with rate $t^{-\frac{1}{p-2}}$, and features extinction in finite time for $1 < p < 2$. For $p = 2$, the solution converges exponentially to zero.

# Stochastic $p$-Laplace: ergodicity



| **Range of $p$** | **Noise** | **Rate** |
|---|---|---|
| $(2,\infty)$ | 0 | $t^{-\frac{1}{p-2}}$ |
| $\{2\}$ | 0 | $e^{-\lambda t}$ |
| $\left[1 \vee \frac{2d}{d+2}, 2\right)$ | 0 | $e^{-\lambda t}$ |
| $\left[1 \vee \left(2-\frac{4}{d}\right), 1 \vee \frac{2d}{d+2}\right) \cap (1,2)$ | 0 | $t^{-\frac{p}{2-p}}$ |
| $(2,\infty)$ | degenerate | $t^{-\frac{1}{p-2}}$ |
| $\{2\}$ | degenerate | $e^{-\lambda t}$ |
| $\left(1 \vee \frac{2d}{d+2}, 2\right) \cap [\sqrt{2}, 2)$ | degenerate | $t^{-\frac{1}{2}}$ |
| $\left(1 \vee \frac{2d}{d+2}, \sqrt{2}\right)$ | degenerate | $t^{-\frac{p^2}{4}}$ |
| $\left[1 \vee \left(2-\frac{4}{d}\right), 2\right) \cap (1,2)$ | degenerate regular | $t^{-\frac{p}{2-p}}$ |
| $(2,\infty)$ | non – degenerate | $e^{-\lambda t}$ |

# Stochastic $p$-Laplace: $\varepsilon$-mixing times

| **Range of $p$** | **Noise** | **Rate** | **Upper bound** |
|---|---|---|---|
| $(2,\infty)$ | 0 | $t^{-\frac{1}{p-2}}$ | $C\varepsilon^{2-p}$ |
| $\{2\}$ | 0 | $e^{-\lambda t}$ | $C+\frac{1}{\lambda}\log(\varepsilon^{-1})$ |
| $\left[1\vee\frac{2d}{d+2},2\right)$ | 0 | $e^{-\lambda t}$ | $C+\frac{1}{\lambda}\log(\varepsilon^{-1})$ |
| $\left[1\vee\left(2-\frac{4}{d}\right),1\vee\frac{2d}{d+2}\right)\cap(1,2)$ | 0 | $t^{-\frac{p}{2-p}}$ | $C\varepsilon^{\frac{2-p}{p}}$ |
| $(2,\infty)$ | degenerate | $t^{-\frac{1}{p-2}}$ | $C\varepsilon^{2-p}$ |
| $\{2\}$ | degenerate | $e^{-\lambda t}$ | $C+\frac{1}{\lambda}\log(\varepsilon^{-1})$ |
| $\left(1\vee\frac{2d}{d+2},2\right)\cap[\sqrt{2},2)$ | degenerate | $t^{-\frac{1}{2}}$ | $C\varepsilon^{-2}$ |
| $\left(1\vee\frac{2d}{d+2},\sqrt{2}\right)$ | degenerate | $t^{-\frac{p^2}{4}}$ | $C\varepsilon^{-\frac{4}{p^2}}$ |
| $\left[1\vee\left(2-\frac{4}{d}\right),2\right)\cap(1,2)$ | degenerate regular | $t^{-\frac{p}{2-p}}$ | $C\varepsilon^{\frac{2-p}{p}}$ |
| $(2,\infty)$ | non $-$ degenerate | $e^{-\lambda t}$ | $C+\frac{1}{\lambda}\log(\varepsilon^{-1})$ |

Using the decay estimates for the deterministic $p$-Laplace, which are known to be optimal, we may obtain **lower bounds for the $\varepsilon$-mixing times**.

Let $\{u_t^x\}$ be the solution to

$$du_t^x = \operatorname{div}\left[|\nabla u_t^x|^{p-2}\nabla u_t^x\right]dt, \quad t>0, \quad u_0^x = x,$$

with Dirichlet boundary conditions on a bounded domain $\mathcal{O}\subset\mathbb{R}^d$.

Let $\mu_*$ be the unique invariant measure of the stochastic $p$-Laplace equation. As we have **additive noise**, we obtain $\mathbb{E}[X_t^x]=\mathbb{E}[u_t^x]$ in $L^2(\mathcal{O})$ for $t\geq 0$. We have that

$$\|u_t^x - \int_{L^2(\mathcal{O})} y\,\mu_*(dy)\|_{L^2(\mathcal{O})} \leq \mathcal{W}_r\left(\operatorname{Law}(X_t^x),\mu_*\right), \quad r\geq 1,$$

Now, as $\|u_t^x\|_{L^2(\mathcal{O})}\to 0$ as $t\to\infty$ and if $\mathcal{W}_r(\operatorname{Law}(X_t^x),\mu_*)\to 0$ as $t\to\infty$ for some $r\geq 1$, we obtain that $\|\int_{L^2(\mathcal{O})} y\,\mu_*(dy)\|_{L^2(\mathcal{O})}=0$, and thus

$$\|u_t^x\|_{L^2(\mathcal{O})} \leq \mathcal{W}_r\left(\operatorname{Law}(X_t^x),\mu_*\right).$$

For example, for $p \in \left(1 \vee \frac{2d}{d+2}, 2\right) \cap [\sqrt{2}, 2)$, with **degenerate noise**, we obtain **the upper and lower bounds for the $\varepsilon$-mixing time** in the $\mathcal{W}_p$-Wasserstein metric

$$\frac{1}{\lambda}\log\left(\frac{c\|x\|_{L^2(\mathcal{O})}}{\varepsilon}\right) \leq \tau^x(\varepsilon) \leq C(\|x\|_{L^2(\mathcal{O})})\varepsilon^{-2}.$$

For $p > 2$ with **degenerate noise**, we obtain that both **the upper bound and the lower bound for the $\varepsilon$-mixing time** are of the same order

$$c(\|x\|_{L^2(\mathcal{O})})\varepsilon^{2-p} \leq \tau^x(\varepsilon) \leq C(\|x\|_{L^2(\mathcal{O})})\varepsilon^{2-p}.$$

# Wrap-up and open problems



- We have proved **ergodicity, mixing, and upper bounds for $\varepsilon$-mixing times** of **locally monotone drift** SPDEs with possibly **degenerate Wiener noise** with **spatial regularity** in the more regular space $V$ under quantitative conditions. The embedding $V \subset H$ does not need to be compact, so we include examples with possibly **unbounded domains** that satisfy a **Poincaré inequality**. We are able to dispense with the **range condition** for the noise, which recovers the PDE result without noise.
- Further examples are the **stochastic power law fluid equations** for $p > 2$, the **stochastic heat equation**, and the **stochastic 1D Burgers** equation.
- Semilinear SPDEs like the **stochastic Allen-Cahn equation**, and *fully locally monotone* SPDEs like the **stochastic Cahn-Hilliard equation** are not covered by our results.
- It would be nice to extend to **Lévy noise**, but so far, we are just able to manage exponentially small jumps.

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**Thank you for your attention!**