

Potentials of occupations measures of Gaussian fields and fractal minimizers

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Riesz kernel $k_\alpha(x) = c(n, \alpha)|x|^{\alpha-n}$, $x \in \mathbb{R}^n \setminus \{0\}$.

In the sequel: $c(n, \alpha) \equiv 1$.

Riesz potential for a nonnegative Radon measure μ ,

$$U^\alpha \mu(x) := \int_{\mathbb{R}^n} k_\alpha(x-y) \mu(dy), \quad x \in \mathbb{R}^n.$$

Mutual Riesz energy for nonnegative Radon measures μ and ν

$$I^\alpha(\mu, \nu) := \int_{\mathbb{R}^n} U^\alpha \mu(x) \nu(dx).$$

Riesz energy $I^\alpha(\mu) := I^\alpha(\mu, \mu)$.

Example: Coulomb interaction energy

Let $n \geq 3$, $\alpha = 2$, so $k_2(x) = |x|^{2-n}$.

Interpret finite discrete subsets $\mu = \{\mu_i\}$ and $\nu = \{\nu_j\}$ as configurations of positively charged particles.

$$I^2(\mu, \nu) = \sum_i \sum_j k_2(\mu_i - \nu_j)$$

is the *mutual Coulomb energy* of the discrete measures $\sum_i \delta_{\mu_i}$ and $\sum_j \delta_{\nu_j}$.

This is the part of the total electrostatic energy of the system caused by the *mutual Coulomb interaction* between μ and ν , self-interactions ignored.

If $\mu = \nu$, then $\sum_i \sum_j 1_{\{i \neq j\}} k_2(\mu_i - \mu_j)$ is the *internal electrostatic self-interaction energy* of a point configuration or a Coulomb gas.

Let $X: [0, 1]^k \rightarrow \mathbb{R}^n$ be a Borel function.

Occupation measure: $\mu_X(B) := \mathcal{L}^k(\{t \in [0, 1]^k : X(t) \in B\})$, $B \subset \mathbb{R}^n$ Borel set.

Occupation time formula:

$$\int_{\mathbb{R}^n} g(x) \mu_X(dx) = \int_{[0,1]^k} g(X(t)) dt, \quad \forall g \in \mathcal{B}_b(\mathbb{R}^n).$$

$X \mapsto \mu_X$ is highly nonlinear.

If $\mu_X \ll \mathcal{L}^n$, then

$$L_X := \frac{d\mu}{d\mathcal{L}^n} \in L^1(\mathbb{R}^n)$$

is called *local time* of X .

For $\gamma \in (0, 1)$ and $X: [0, 1]^k \rightarrow \mathbb{R}^n$, define $X \in C_0^\gamma([0, 1]^k; \mathbb{R}^n)$ if $X(0) = 0$ and

$$\|X\|_{C_0^\gamma} := \sup_{s \neq t} \frac{|X(t) - X(s)|}{|t - s|^\gamma} < \infty.$$

Set

$$\mathcal{B}_\varrho^\gamma := \{X \in C_0^\gamma([0, 1]^k; \mathbb{R}^n) : \|X\|_{C_0^\gamma} \leq \varrho\}.$$

1.1 Hölder constrained minimal self-interaction

Recall that

$$X \mapsto I^\alpha(\mu_X) = \int_{[0,1]^k} \int_{[0,1]^k} |X(t) - X(s)|^{\alpha-n} ds dt.$$

Note that $\mu \mapsto I^\alpha(\mu)$ is quadratic and convex, but $X \mapsto I^\alpha(\mu_X)$ is not convex. Not even the effective domain $\{X: [0, 1]^k \rightarrow \mathbb{R}^n : I^\alpha(\mu_X) < \infty\}$ is convex.

Theorem 1. (Hinz, T., Viitasaari, 2025+) Let $0 < \alpha < n$, $0 < \gamma < \frac{k}{n-\alpha} \wedge 1$.

- i. For any $\varrho > 0$, there is some $X^* \in \mathcal{B}_\varrho^\gamma$ minimizing $X \mapsto I^\alpha(\mu_X)$ over $\mathcal{B}_\varrho^\gamma$.
- ii. For any minimizer X^* of $X \mapsto I^\alpha(\mu_X)$ in $\mathcal{B}_\varrho^\gamma$ and any rectangle $\mathcal{R} \subset [0, 1]^k$, the image $X^*(\mathcal{R})$ satisfies

$$\mathcal{H}^{n-\alpha}(X^*(\mathcal{R})) = \infty, \quad \mathcal{H}^{\frac{k}{\gamma} \wedge n}(X^*(\mathcal{R})) < \infty$$

and, as a consequence,

$$n - \alpha \leq \dim_H X^*(\mathcal{R}) \leq \frac{k}{\gamma} \wedge n.$$

If $1 \leq k \leq n$, $\dim_H X^*(\mathcal{R})$ can be arbitrary close to $n - \alpha$, provided γ is close enough to $\frac{k}{n - \alpha} \wedge 1$. If α is non-integer, it forces $\dim_H X^*(\mathcal{R})$ to be non-integer.

Proof of (ii):

$\mathcal{H}^{n-\alpha}(X^*(\mathcal{R})) = \infty$ follows from $I^\alpha(\mu_{X^*}) < \infty$.

$\mathcal{H}^{\frac{k}{n-\alpha} \wedge n}(X^*(\mathcal{R})) < \infty$ follows from $X^* \in \mathcal{B}_\rho^\gamma$.

(See Falconer [Fal90]).

For the proof of (i), we shall need two lemmas.

Lemma 2. *Every sequence $(X_i)_i \in \mathcal{B}_\rho^\gamma$ has a subsequence $(X_{i_j})_j$ converging uniformly to some $X^* \in \mathcal{B}_\rho^\gamma$ such that $(\mu_{X_{i_j}})_j$ converges weakly to μ_{X^*} .*

Proof. The first statement follows from the Arzelà-Ascoli Theorem.

Since $\mathcal{L}^k([0, 1]^k) = 1$, the uniform convergence implies convergence in probability $\mathcal{L}^k|_{[0,1]^k}$. Let $\varphi \in C_c(\mathbb{R}^n)$, by uniform continuity, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|\varphi(x) - \varphi(y)| < \varepsilon$. By the occupation time formula

$$\begin{aligned} & \left| \int_{\mathbb{R}^n} \varphi(x) \mu_{X_{i_j}}(dx) - \int_{\mathbb{R}^n} \varphi(x) \mu_{X^*}(dx) \right| \leq \int_{[0,1]^k} |\varphi(X_{i_j}(t)) - \varphi(X^*(t))| dt \\ & \leq \int_{|X_{i_j} - X^*| > \frac{\delta}{2}} |\varphi(X_{i_j}(t)) - \varphi(X^*(t))| dt + \int_{|X_{i_j} - X^*| \leq \frac{\delta}{2}} |\varphi(X_{i_j}(t)) - \varphi(X^*(t))| dt \\ & \leq 2\|\varphi\|_\infty \mathcal{L}^k|_{[0,1]^k} \left(|X_{i_j} - X^*| > \frac{\delta}{2} \right) + \varepsilon \mathcal{L}^n(\text{supp}(\varphi)). \end{aligned}$$

By a density argument, $(\mu_{X_{i_j}})_j$ converges weakly to μ_{X^*} . □

A centered Gaussian random field $b^H: [0, \infty)^k \times \Omega \rightarrow \mathbb{R}$ with $b^H(0) = 0$ \mathbb{P} -a.s. and

$$\mathbb{E}[|b^H(t) - b^H(s)|^2] = |t - s|^{2H}, \quad s, t \in [0, \infty)^k$$

over a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, is called a *fractional Brownian $(k, 1)$ -field* with *Hurst index* $H \in (0, 1)$.

$k = 1$: fractional Brownian motion (fBm)

$H = \frac{1}{2}$: Lévy's Brownian field

$k = 1, H = \frac{1}{2}$: Brownian motion (Bm)

Note that for $\ell \in \mathbb{N}$, ℓ even

$$\mathbb{E}[|b^H(t) - b^H(s)|^\ell] = (\ell - 1)!! |t - s|^{\ell H}, \quad s, t \in [0, \infty)^k.$$

Let b_i^H , $1 \leq i \leq n$ be independent fractional Brownian $(k, 1)$ -fields with Hurst index $H \in (0, 1)$. Then

$$B^H := (b_1^H, \dots, b_n^H): [0, \infty)^k \times \Omega \rightarrow \mathbb{R}^n,$$

is a centered Gaussian field with $B^H(0) = 0$ \mathbb{P} -a.s. and

$$\mathbb{E}[|B^H(t) - B^H(s)|^\ell] = n^{\frac{\ell}{2}} \mathbb{E}[|b^H(t) - b^H(s)|^\ell], \quad s, t \in [0, \infty)^k$$

for $\ell \in \mathbb{N}$, ℓ even, called *fractional Brownian (k, n) -field with Hurst index H* .

Lemma 3. *Let $0 < \alpha < n$, $0 < \gamma < \frac{k}{n-\alpha} \wedge 1$, $\varrho > 0$. There exists $X \in \mathcal{B}_\varrho^\gamma$ such that $I^\alpha(\mu_X) < \infty$.*

Proof. By the Kolmogorov-Chentsov theorem there exists a modification of B^H (denoted by the same symbol) and a random variable $K > 0$ and an event $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}(\Omega_0) = 1$ such that

$$|B^H(t, \omega) - B^H(s, \omega)| \leq K(\omega) |t - s|^\gamma, \quad s, t \in [0, 1]^k, \quad \omega \in \Omega_0.$$

For some $c, c' > 0$,

$$\begin{aligned} \mathbb{E}[|B^H(t) - B^H(s)|^{\alpha-n}] &= c |t - s|^{-nH} \int_{\mathbb{R}^n} \exp\left(-\frac{|y|^2}{2|t - s|^{2H}}\right) |y|^{\alpha-n} dy \\ &= c' |t - s|^{(\alpha-n)H} \end{aligned}$$

Since $(n - \alpha)H < k$

$$\begin{aligned} & \mathbb{E} \int_{[0,1]^k} \int_{\{s \in [0,1]^k : |B^H(t) - B^H(s)| < R\}} |B^H(t) - B^H(s)|^{\alpha-n} dt ds \\ &= c' \int_{[0,1]^k} \int_{\{s \in [0,1]^k : |B^H(t) - B^H(s)| < R\}} |t - s|^{(\alpha-n)H} ds dt < \infty \end{aligned}$$

For $|z| \geq R$, $k_\alpha(z) \leq R^{\alpha-n}$, and hence there exists $\Omega_1 \in \mathcal{F}$ with $\Omega_1 \subset \Omega_0$ and $\mathbb{P}(\Omega_1) = 1$ such that

$$I^\alpha(\mu_{B^H(\cdot, \omega)}) < \infty \quad \forall \omega \in \Omega_1.$$

For $\omega \in \Omega_1$, set

$$X(t) := \varrho(1 + K(\omega))^{-1} B^H(t, \omega), \quad t \in [0, 1]^k,$$

then $X \in \mathcal{B}_\varrho^\gamma$ and $I^\alpha(\mu_X) < \infty$.

□

Proof of Theorem 1. (The direct method in the calculus of variations).

Note that $\inf_{X \in \mathcal{B}_\rho^\gamma} I^\alpha(\mu_X) < \infty$ by Lemma 3. If $(X_i)_i \subset \mathcal{B}_\rho^\gamma$ is such that

$$\lim_{i \rightarrow \infty} I^\alpha(\mu_{X_i}) = \inf_{X \in \mathcal{B}_\rho^\gamma} I^\alpha(\mu_X),$$

then, as $\mu \mapsto I^\alpha(\mu)$ is lower semi-continuous with respect to weak convergence, we have for X^* as in Lemma 1 that

$$\inf_{X \in \mathcal{B}_\rho^\gamma} I^\alpha(\mu_X) \leq I^\alpha(\mu_{X^*}) \leq \liminf_{j \rightarrow \infty} I^\alpha(\mu_{X_{i_j}}) = \inf_{X \in \mathcal{B}_\rho^\gamma} I^\alpha(\mu_X).$$

□

- X^* is not expected to be unique. The radial symmetry of k_α produces easy counterexamples.
- For $K \subset \mathbb{R}^n$, K compact, $\mu \mapsto I^\alpha(\mu)$ admits a unique minimizer in the class $\mathcal{M}^1(K)$, called equilibrium measure μ_K of K with respect to I^α . For $K = \overline{B}(0, \rho k^{\frac{\gamma}{2}})$, $I^\alpha(\mu_K) \leq I^\alpha(\mu_{X^*})$. However, μ_K is not an occupation measure in general, while μ_{X^*} has to satisfy $\text{supp } \mu_{X^*} = X^*([0, 1]^k)$.
- The **general philosophy** is that uniform convergence of the processes preserves **regularity**, whereas weak convergence of the occupation measures preserves **irregularity**.

1.2 Doubly constrained minimal interaction

Let ν be a compactly supported Borel probability measure on \mathbb{R}^n (environment / medium). Consider

$$\min \left\{ X \mapsto I^\alpha(\mu_X, \nu) = \int_{\mathbb{R}^n} \int_{[0,1]^k} |X(t) - y|^{\alpha-n} dt \nu(dy) \right\}$$

(“mutual interaction energy” in “medium” ν , if $n=2$, $k=1$: “path in minefield”).

Set

$$\mathcal{K} := \mathcal{K}(\gamma, \alpha, \varrho, M) := \left\{ X \in \mathcal{B}_\varrho^\gamma : \sup_{x \in \text{supp } \nu} U^\alpha \mu_X(x) \leq M \right\}.$$

(If $k=1$: Hölder curves “whose velocity does not vary too wildly”).

Remark. We need that $\text{supp } \nu \cap \text{supp } \mu_X \neq \emptyset$. Note that $\text{supp } \mu_X \subset \overline{B}(0, \varrho k^{\frac{\gamma}{2}})$ is compact for $X \in \mathcal{B}_\varrho^\gamma$.

Theorem 4. (Hinz, T., Viitasaari, 2025+) Let $0 < \alpha < n$, $0 < \gamma < \frac{k}{n-\alpha} \wedge 1$, $\varrho > 0$, $M > 0$.

- i. If $\mathcal{K} \neq \emptyset$, $\exists X^* \in \mathcal{K}$ minimizing $X \mapsto I^\alpha(\mu_X, \nu)$.
- ii. For any minimizer of $X \mapsto I^\alpha(\mu_X, \nu)$ and any rectangle $\mathcal{R} \subset [0, 1]^k$, the image $X^*(\mathcal{R})$ satisfies

$$\mathcal{H}^{n-\alpha}(X^*(\mathcal{R})) = \infty, \quad \mathcal{H}^{\frac{k}{\gamma} \wedge n}(X^*(\mathcal{R})) < \infty$$

and, as a consequence,

$$n - \alpha \leq \dim_H X^*(\mathcal{R}) \leq \frac{k}{\gamma} \wedge n.$$

- iii. $\exists \varrho_1 > 0$ and $\exists M_1 > 0$ depending on γ, α, ν , such that $\mathcal{K} \neq \emptyset$, whenever $\varrho > \varrho_1$ and $M > M_1$.

Proof of Theorem 4. (Sketch).

- i. As $y \mapsto U^\alpha \nu(y)$ is lower semi-continuous and nonnegative [Lan72], it follows that $X \mapsto I^\alpha(\mu_X, \nu)$ is lower semi-continuous in $L^1([0, 1]^k; \mathbb{R}^n)$. Furthermore, it follows from properties of U^α that \mathcal{K} is compact in L^1 . The proof can be completed by the direct method in the calculus of variations, see [JL98].
- ii. As before.
- iii. We need to construct such an element in \mathcal{K} , see Lemma 6 below.

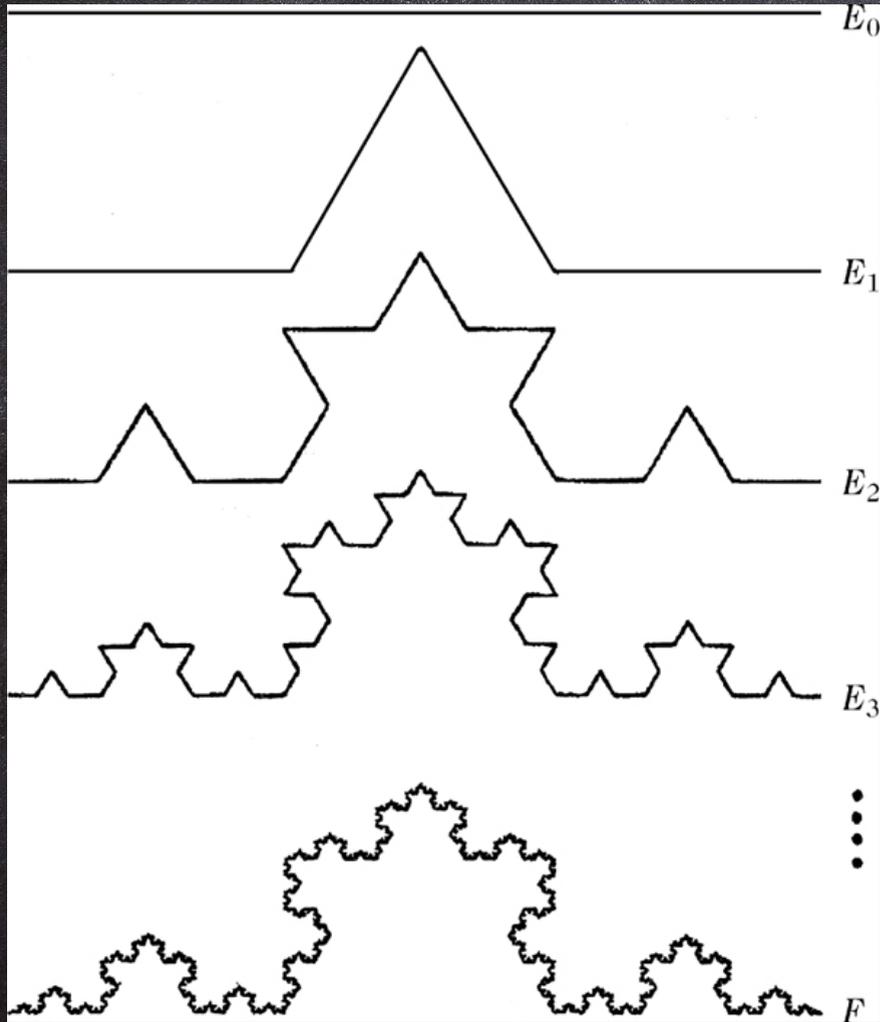
Alternative for (iii): Use Assouad's embedding theorem [Ass83] if

$$k \left(\left\lfloor \frac{1}{\gamma} \right\rfloor + 1 \right) \leq n \leq \frac{k}{\gamma} + \alpha,$$

which implies that $\exists c, C > 0$ with

$$c|t - s|^\gamma \leq |X(t) - X(s)| \leq C|t - s|^\gamma.$$

Example: Koch curve $\gamma = \frac{\log 3}{\log 4}$.



Item (iii) of the previous proof has a probabilistic proof.

If $X: [0, 1]^k \rightarrow \mathbb{R}^n$ has local times $L_X \in L_{\text{loc}}^p$ and $n - \alpha < \frac{n}{q}$ with $\frac{1}{p} + \frac{1}{q} = 1$, then there exists $C > 0$ with $U^\alpha \mu_X \leq C$. If L_X is locally bounded, then $U^\alpha \mu_X$ is locally bounded for $0 < \alpha < n$.

We claim that potentials of occupation measures are good replacements for L_X .

For technical reasons, replace $U^\alpha \mu_X$ by $G^\alpha \mu_X$, that is, G^α is the *Bessel potential*, i.e.

$$G^\alpha \mu(x) = \int_{\mathbb{R}^n} g_\alpha(x - y) \mu(dy), \quad x \in \mathbb{R}^n,$$

where the *Bessel kernel* g_α is defined via $\hat{g}_\alpha(\xi) = \left[(1 + |\xi|^2)^{\frac{1}{2}} \right]^{-\alpha}$. U^α and G^α are comparable.

Let $X: [0, 1]^k \times \Omega \rightarrow \mathbb{R}^n$ be a centered Gaussian random field, $X(0) = 0$ \mathbb{P} -a.s. with $\sigma^2(s, t) := \text{Cov}(X(t) - X(s))$.

We say that X is *locally non-deterministic* if $\forall \ell \geq 2, \exists c_\ell > 0, \exists \varepsilon_\ell > 0$ such that

$$\text{Var} \left(\sum_{j=1}^{\ell} \langle u_j, \sigma_j^{-1}(X(t_j) - X(t_{j-1})) \rangle \right) \geq c_\ell \sum_{j=1}^{\ell} |u_j|^2,$$

for every $u_1, \dots, u_\ell \in \mathbb{R}^n$ and for a.e. $t_1, \dots, t_\ell \in ([0, 1]^k)^\ell$ lying in a single subcube of side length ε_ℓ and such that $|t_{j+1} - t_j| \leq |t_{j+1} - t_i|$ for all $1 \leq i \leq j \leq \ell$. Here, $t_0 := 0$. Compare with [GH80].

Theorem 5. (Hinz, T., Viitasaari, 2025+, for L_X , see [Pit78]) Let X be as before, and assume that X is locally non-deterministic. Let $\varphi: [0, 1]^k \rightarrow \mathbb{R}^n$ be a bounded Borel function. Assume that $\det \sigma^2(s, t) > 0$. Let $0 < \alpha < n$, and assume that there exists $\delta > 0$ with

$$\operatorname{esssup}_{s \in [0, 1]^k} \int_{[0, 1]^k} \frac{dt}{|\det \sigma^2(s, t)|^{\frac{n-\alpha}{2n} + \delta}} < \infty.$$

Then $G^\alpha \mu_{X+\varphi}$ is locally Hölder continuous.

Lemma 6. *Let B^H be a fractional Brownian (k, n) -field with Hurst index $H \in (0, 1)$, let $\varphi: [0, 1]^k \rightarrow \mathbb{R}^n$ be a bounded Borel function.*

- i. If $n - \frac{k}{H} < \alpha < n$ and $0 < \gamma < H$, there exists $\Omega_1 \in \mathcal{F}$ with $\mathbb{P}(\Omega_1) = 1$ such that $B^H(\cdot, \omega) \in C_0^\gamma([0, 1]^k; \mathbb{R}^n)$ and $G^\alpha \mu_{B^H(\cdot, \omega) + \varphi} \in C(\mathbb{R}^n)$ as well as $G^\alpha \mu_{B^H(\cdot, \omega) + \varphi}$ and $U^\alpha \mu_{B^H(\cdot, \omega) + \varphi}$ are bounded for all $\omega \in \Omega_1$.*
- ii. If $n < \frac{k}{H}$ and $0 < \gamma < H$ there exists $\Omega_1 \in \mathcal{F}$ with $\mathbb{P}(\Omega_1) = 1$ such that $B^H(\cdot, \omega) \in C_0^\gamma([0, 1]^k; \mathbb{R}^n)$ and $B^H(\cdot, \omega) + \varphi$ has locally bounded local times $L_{B(\cdot, \omega) + \varphi} \in C(\mathbb{R}^n)$ for all $\omega \in \Omega_1$.*

Proof. The hypotheses of Theorem 5 are satisfied, so we can apply the Kolmogorov-Chentsov theorem. □

Here, the situation is **typical**, in contrast to the Assouad embedding theorem.

We have used **local non-determinism** to obtain **regularity** for $U^\alpha \mu_X$, $G^\alpha \mu_X$, which encodes the **irregularity** of X and implies certain **lower bounds** for X in terms of variation, see our preprint [arXiv:2512.14248](https://arxiv.org/abs/2512.14248).

In the future, we would like to use potentials as replacements for L_X to obtain **lower bounds** for the paths. This is a publication in preparation with Hinz, T. and Viirtasaari.

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Thank you for your attention!